

SOMMERS DUALITY AND THE SPRINGER–STEINBERG MAP FOR CANONICAL LEFT CELLS IN TYPES B AND C

ABSTRACT. Let $(G, G^\vee)$ be the Langlands-dual pairs formed by $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$, and $\mathcal{O} \subset \mathrm{Lie}(G)$ be a special nilpotent orbit whose Barbasch–Vogan dual orbit $\mathcal{O}^\vee \subset \mathrm{Lie}(G^\vee)$ has good parity. We prove that Sommers’ generalized duality map for $\mathcal{O}$ agrees with the Springer–Steinberg map on the Weyl-group elements parametrizing special-unipotent representations attached to $\mathcal{O}^\vee$ in the work of Barbasch–Vogan.

1. INTRODUCTION

Let $G^\vee$ be a connected complex reductive group, $T^\vee \subseteq B^\vee$ be a maximal torus contained in a Borel subgroup, and write

$$\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee), \quad W = N_{G^\vee}(T^\vee)/T^\vee.$$

Let $\mathfrak{n}^\vee$ be the nilradical of $\mathrm{Lie}(B^\vee)$. For $v \in W$, the **Springer–Steinberg orbit** $\mathrm{St}_{G^\vee}(v)$ is the unique nilpotent $G^\vee$ -orbit whose intersection with

$$\mathfrak{n}^\vee \cap \mathrm{Ad}(\dot{v})\mathfrak{n}^\vee$$

is dense and open in this vector space, where $\dot{v} \in N_{G^\vee}(T^\vee)$ represents v . This defines the Springer–Steinberg map; see Definition 2.3 and [19, 12].

A well-known special case arises from an even nilpotent orbit $\mathcal{O}^\vee$. Choose a Jacobson–Morozov triple (e, h, f) for $\mathcal{O}^\vee$ with $h \in \mathrm{Lie}(T^\vee)$ dominant and the eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}^\vee$ are all even. Let $L^\vee = Z_{G^\vee}(h)$ be the Dynkin Levi subgroup, let $W_I \subseteq W$ be its Weyl group, and let w_I be the longest element of W_I . Then

$$(1) \quad \mathrm{St}_{G^\vee}(w_I) = \mathcal{O}^\vee.$$

In this manuscript, we generalize (1) when $(G, G^\vee)$ is one of the ordered pairs

$$(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{SO}_{2n+1}(\mathbb{C})), \quad (\mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})).$$

Assume that $\mathcal{O}^\vee$ has **good parity** in the sense of [4, Definition 2.6] (see Definition 4.1), and put $\mathcal{O} = d_{BV}(\mathcal{O}^\vee) \subseteq \mathfrak{g}$ be the Barbasch–Vogan dual orbit of $\mathcal{O}^\vee$ (c.f. [6], [8, §6.3]). For $\xi \in \mathcal{O}$, let $A(\mathcal{O}) = Z_G(\xi)/Z_G(\xi)^\circ$, and let $\overline{A}(\mathcal{O})$ be Lusztig’s canonical quotient. In [18], Sommers generalized the Barbasch–Vogan dual map by assigning each conjugacy class $[x]$ in $\overline{A}(\mathcal{O})$ a nilpotent orbit:

$$d_S(\mathcal{O}, [x]) = \mathcal{O}_{[x]}^\vee \subseteq \mathfrak{g}^\vee.$$

In particular, when $[x] = [0]$ is trivial, then one obtains the usual Barbasch–Vogan dual $d_S(\mathcal{O}, 0) = d_{BV}(\mathcal{O}) = \mathcal{O}^\vee$.

Let Γ be the equal-parameter Kazhdan–Lusztig left cell containing w_I , with the conventions of Section 2. By [6, §5], all special-unipotent representations of G attached to the orbit $\mathcal{O}^\vee$ in (1) are parametrized by the set $\Gamma \cap \Gamma^{-1}$ which, in turn, is bijective to the set of conjugacy classes of $\overline{A}(\mathcal{O})$. Then the main result of the manuscript is the following:

Theorem 1.1. *With the notation above, the cell-constituent labeling of Definition 8.5 gives a bijection*

$$\overline{A}(\mathcal{O}) \longrightarrow \Gamma \cap \Gamma^{-1}, \quad x \longmapsto v_x.$$

It satisfies $v_0 = w_I$ and

$$(2) \quad \mathrm{St}_{G^\vee}(v_x) = d_S(\mathcal{O}, x) \quad (x \in \overline{A}(\mathcal{O})).$$

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Here we identify each element of the abelian group $\overline{A}(\mathcal{O})$ with its singleton conjugacy class.

For the groups studied in this manuscript, the theorem above extends the result of Barbasch–Trapa [5, Theorem 1.1], where they studied the stable sums of special unipotent representations under the assumption that both $\mathcal{O}$ and $\mathcal{O}^\vee$ are even. Under their hypotheses, the orbits $d_S(\mathcal{O}, x)$ exhaust the special piece (see (12)) of $\mathcal{O}^\vee$, and our correspondence agrees with theirs. Our result does not require the evenness assumption on $\mathcal{O}$ and, in such a case, some Sommers duals may lie outside this special piece. We expect that these results may also be relevant to covering versions of Barbasch–Vogan duality for complex groups; see [20, Remark 3.13].

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2. GEOMETRIC AND ALGEBRAIC CONVENTIONS

In this section and Section 3, let H be a connected complex reductive group, with a Borel subgroup B and a maximal torus $T \subseteq B$. Write $\mathfrak{h} = \text{Lie}(H)$, let $\mathfrak{n}$ be the nilradical of $\text{Lie}(B)$, and let $\mathcal{N}$ be the nilpotent cone. Put $\mathcal{B} = H/B$, $d = \dim \mathcal{B}$, and $W = N_H(T)/T$. Let S_0 be the simple reflections, ℓ the length function, and $\text{sgn}(w) = (-1)^{\ell(w)}$.

Identify $\mathcal{B}$ with the variety of Borel subgroups of H , and write $\mathfrak{n}_{B'}$ for the nilradical attached to $B' \in \mathcal{B}$. The Steinberg variety and its projection to $\mathcal{N}$ are

$$Z = \{(\xi, B_1, B_2) \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} : \xi \in \mathfrak{n}_{B_1} \cap \mathfrak{n}_{B_2}\}, \quad \text{pr}(\xi, B_1, B_2) = \xi.$$

For $v \in W$, let Z_v be the closure of the locus where (B_1, B_2) has relative position v . These are precisely the irreducible components of Z , each of complex dimension $2d$; they are the closures of the conormal bundles to the diagonal H -orbits in $\mathcal{B} \times \mathcal{B}$.

Definition 2.1 (Conormal class, see [12] §4.1). We use the ordinary Springer convention: the regular nilpotent orbit with trivial local system corresponds to the trivial representation of W . Let

$$\iota : H_{4d}^{\text{BM}}(Z, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}[W]$$

be the Springer convolution-algebra isomorphism in this convention, normalized by $\iota([Z_s]) = s - 1$ for $s \in S_0$. Here Borel–Moore homology is in real degree $4d$, and $[Z_v]$ denotes the fundamental class, mapped to the homology of Z . The conormal classes

$$D_v = \iota([Z_v]) \quad (v \in W)$$

form a basis of $\mathbb{C}[W]$.

Lemma 2.2. *For every $v \in W$, the variety $\text{pr}(Z_v)$ is the closure of a unique nilpotent H -orbit $\mathcal{O}$. Moreover, $\mathcal{O} \cap (\mathfrak{n} \cap \text{Ad}(v)\mathfrak{n})$ is dense and open in $\mathfrak{n} \cap \text{Ad}(v)\mathfrak{n}$.*

Proof. The projection is proper because $\mathcal{B} \times \mathcal{B}$ is projective. Thus $\text{pr}(Z_v)$ is closed, irreducible, and H -stable; since there are finitely many nilpotent orbits, it has a unique open dense orbit $\mathcal{O}$. The relative-position- v locus is dense in Z_v and has image $H \cdot (\mathfrak{n} \cap \text{Ad}(v)\mathfrak{n})$, so this image meets $\mathcal{O}$. Hence $\mathcal{O} \cap (\mathfrak{n} \cap \text{Ad}(v)\mathfrak{n})$ is nonempty and open in the irreducible vector space $\mathfrak{n} \cap \text{Ad}(v)\mathfrak{n}$, and therefore dense. $\square$

Definition 2.3 (Springer–Steinberg map). The Springer–Steinberg map sends $v \in W$ to the unique dense orbit $\text{St}_H(v)$ in $\text{pr}(Z_v)$. Thus

$$\text{St}_H : W \longrightarrow \{\text{nilpotent } H\text{-orbits in } \mathfrak{h}\}, \quad \text{pr}(Z_v) = \overline{\text{St}_H(v)}.$$

Definition 2.4 (Ordinary Springer support). For an irreducible W -representation R , let $(\mathcal{O}_R, \mathcal{L}_R)$ be its pair under the ordinary Springer correspondence, where $\mathcal{L}_R$ is an irreducible H -equivariant local system on $\mathcal{O}_R$. We write

$$R \simeq \text{Spr}_H(\mathcal{O}_R, \mathcal{L}_R), \quad \text{Supp}_H(R) = \mathcal{O}_R.$$

We use the following specialization of the equal-parameter Kazhdan–Lusztig basis:

$$a_v = \sum_{y \leq v} (-1)^{\ell(v) + \ell(y)} P_{y,v}(1) y \quad (v \in W),$$

where $\leq$ is Bruhat order and $P_{y,v}$ is the Kazhdan–Lusztig polynomial. In particular, $a_s = s - 1 = D_s$. Write $\leq_L$, $\leq_R$, and $\leq_{LR}$ for the standard left, right, and two-sided cell preorders, oriented so that the spans of $\{a_y : y \leq_L v\}$ are left ideals. We use the usual cell subquotients; see [6, (3.11)–(3.19)], where the cones are written with the opposite inequality. Explicitly, for a left cell Γ , put

$$L_{\leq \Gamma} = \text{span}_{\mathbb{C}}\{a_y : y \leq_L \Gamma\}, \quad L_{< \Gamma} = \text{span}_{\mathbb{C}}\{a_y : y \leq_L \Gamma, y \notin \Gamma\}, \quad [\Gamma] = L_{\leq \Gamma} / L_{< \Gamma}.$$

Here $y \leq_L \Gamma$ means $y \leq_L v$ for some $v \in \Gamma$; we use the analogous notation for two-sided cells. The descent sets are

$$\text{Des}_L(v) = \{s \in S_0 : sv < v\}, \quad \text{Des}_R(v) = \{s \in S_0 : vs < v\}.$$

Definition 2.5 (Family algebra and central projection). For a two-sided cell $\mathfrak{d}$, put

$$J_{\leq \mathfrak{d}} = \text{span}_{\mathbb{C}}\{a_y : y \leq_{LR} \mathfrak{d}\}, \quad J_{< \mathfrak{d}} = \text{span}_{\mathbb{C}}\{a_y : y \leq_{LR} \mathfrak{d}, y \notin \mathfrak{d}\}.$$

These are two-sided ideals. The irreducible constituents of $J_{\leq \mathfrak{d}} / J_{< \mathfrak{d}}$ form the family $\mathcal{F}_{\mathfrak{d}}$ associated with $\mathfrak{d}$ [13]. Under the Wedderburn decomposition $\mathbb{C}[W] \simeq \bigoplus_{R \in \text{Irr}(W)} \text{End}_{\mathbb{C}}(R)$, set

$$A_{\mathfrak{d}} = \bigoplus_{R \in \mathcal{F}_{\mathfrak{d}}} \text{End}_{\mathbb{C}}(R), \quad p_{\mathfrak{d}} : \mathbb{C}[W] \longrightarrow A_{\mathfrak{d}}, \quad a \longmapsto (R(a))_{R \in \mathcal{F}_{\mathfrak{d}}}.$$

We identify $A_{\mathfrak{d}}$ with its two-sided ideal in $\mathbb{C}[W]$. Thus $p_{\mathfrak{d}}(a) = z_{\mathfrak{d}}a = az_{\mathfrak{d}}$, where $z_{\mathfrak{d}}$ is the central idempotent acting as the identity on the representations in $\mathcal{F}_{\mathfrak{d}}$ and as zero on all others.

The following standard facts are the comparison results needed below. We state them in our conventions to distinguish the conormal classes from the Kazhdan–Lusztig basis.

Proposition 2.6 (Conormal and cell comparison). (i) *For every $v \in W$,*

$$(3) \quad a_v = \sum_{y \leq v} m_{y,v} D_y, \quad m_{y,v} \in \mathbb{Z}_{\geq 0}, \quad m_{v,v} = 1.$$

For $s \in S_0$, the descent relations are

$$\begin{aligned} D_v s &= -D_v, & a_v s &= -a_v, & \text{if } vs < v, \\ s D_v &= -D_v, & s a_v &= -a_v, & \text{if } sv < v. \end{aligned}$$

(ii) *Central projection induces an algebra isomorphism*

$$(4) \quad J_{\leq \mathfrak{d}} / J_{< \mathfrak{d}} \xrightarrow{\sim} A_{\mathfrak{d}}.$$

Proof. In the notation of [12], $a_v = a(v)$ and $D_v = b(v) = c(v)$. Thus (3) follows from [12, Lemmas 1(ii) and 2, Theorem 6]. The right descent relation for D_v is [12, Proposition 3(i)]; exchanging the two flags gives the left relation. The relations for a_v are the usual Kazhdan–Lusztig multiplication formulas.

For part (ii), semisimplicity of $\mathbb{C}[W]$ shows that the simple ideals occurring in $J_{\leq \mathfrak{d}}$ but not in $J_{< \mathfrak{d}}$ are exactly those indexed by $\mathcal{F}_{\mathfrak{d}}$. Projection onto their sum therefore has kernel $J_{< \mathfrak{d}}$ and image $A_{\mathfrak{d}}$. $\square$

For a closed H -stable subset $X \subseteq \mathcal{N}$, define

$$F_X = \text{span}_{\mathbb{C}}\{D_v : \text{St}_H(v) \subseteq X\}.$$

Geometrically, this is the image under ι of the top Borel–Moore homology of $\text{pr}^{-1}(X)$.

Proposition 2.7 (The orbit filtration). *For every closed H -stable subset $X \subseteq \mathcal{N}$,*

$$(5) \quad F_X = \bigoplus_{\text{Supp}_H(R) \subseteq X} \text{End}_{\mathbb{C}}(R).$$

In particular,

$$(6) \quad R(D_v) \neq 0 \implies \text{Supp}_H(R) \subseteq \overline{\text{St}_H(v)}$$

for every $v \in W$ and every irreducible W -representation R .

Proof. For $X = \overline{\mathcal{O}}$, [12, Proposition 4] identifies F_X as a $W \times W$ -submodule and decomposes it into the Springer representations attached to orbits in X . Under the Wedderburn decomposition this is exactly (5). Finally, $D_v \in F_{\overline{\text{St}_H(v)}}$, which gives (6). $\square$

3. A PARABOLIC ORBIT COMPARISON CRITERION

Fix $I \subseteq S_0$. Let W_I be the subgroup generated by I whose longest element is denoted as w_I . Also, let Γ be the left cell containing w_I , and $\mathfrak{d}$ be the two-sided cell containing Γ . Put

$$e = \frac{1}{|W_I|} \sum_{u \in W_I} (-1)^{\ell(u)} u, \quad Q_I = \{v \in W : I \subseteq \text{Des}_L(v) \cap \text{Des}_R(v)\}.$$

The element e is the parabolic sign idempotent: it satisfies $e^2 = e$ and $ue = eu = \text{sgn}(u)e$ for every $u \in W_I$. The set Q_I consists of the unique maximal-length representatives of the double cosets in $W_I \backslash W / W_I$.

Lemma 3.1 (Parabolic bases and their projections). *The two families*

$$\{a_v : v \in Q_I\}, \quad \{D_v : v \in Q_I\}$$

are bases of $e\mathbb{C}[W]e$. Moreover,

$$Q_I \cap \mathfrak{d} = \Gamma \cap \Gamma^{-1} =: \mathcal{S},$$

and the elements $p_{\mathfrak{d}}(D_v)$, $v \in \mathcal{S}$, form a basis of $eA_{\mathfrak{d}}e$. The projection $p_{\mathfrak{d}}$ also induces a W -module isomorphism $[\Gamma] \simeq A_{\mathfrak{d}}e$.

Proof. For $b \in \mathbb{C}[W]$, averaging over W_I gives

$$\begin{aligned} b \in \mathbb{C}[W]e &\iff be = b \iff bu = \text{sgn}(u)b \quad (u \in W_I), \\ b \in e\mathbb{C}[W] &\iff eb = b \iff ub = \text{sgn}(u)b \quad (u \in W_I). \end{aligned}$$

Thus $\mathbb{C}[W]e$, under left multiplication, is isomorphic to $\text{Ind}_{W_I}^W(\text{sgn})$. If $I \subseteq \text{Des}_R(v)$, the descent relations give $a_v s = -a_v$ for every $s \in I$, and hence $a_v e = a_v$. There is exactly one such v in each right coset of W_I , namely its maximal-length representative. Since $|W_I \backslash W| = \dim \mathbb{C}[W]e$, the set $\{a_v \mid I \subseteq \text{Des}_R(v)\}$ therefore form a basis of $\mathbb{C}[W]e$.

Similarly, if $v \in Q_I$, the left and right descent relations give $ea_v e = a_v$ and $eD_v e = D_v$. On each double coset $W_I v W_I$, the simultaneous left and right sign conditions determine all coefficients of an element of $e\mathbb{C}[W]e$ from one coefficient. The signed sum

$$\sum_{w \in W_I v W_I} (-1)^{\ell(w)} w$$

is nonzero and satisfies both conditions. Consequently,

$$\dim e\mathbb{C}[W]e = |W_I \backslash W / W_I| = |Q_I|.$$

Both $\{a_v : v \in Q_I\}$ and $\{D_v : v \in Q_I\}$ are linearly independent subsets of bases of $\mathbb{C}[W]$ and have this cardinality. Thus both are bases of $e\mathbb{C}[W]e$.

We next determine which indices survive projection to the family of $\mathfrak{d}$. Since $P_{u, w_I} = 1$ for $u \in W_I$,

$$a_{w_I} = (-1)^{\ell(w_I)} |W_I| e.$$

Consequently, if $I \subseteq \text{Des}_R(v)$, then

$$a_v \in \mathbb{C}[W]e = \mathbb{C}[W]a_{w_I} \subseteq L_{\leq \Gamma}, \quad \text{hence } v \leq_L w_I.$$

If also $v \in \mathfrak{d}$, then $v \sim_L w_I$ and hence $v \in \Gamma$. Conversely, every $v \in \Gamma$ has the same right descent set as w_I , namely I . Thus the indices of the one-sided basis in $\mathfrak{d}$ are exactly Γ . Since taking inverse interchanges left and right cells and fixes w_I , the corresponding indices with $I \subseteq \text{Des}_L(v)$ are exactly Γ^{-1} . Taking the intersection proves

$$Q_I \cap \mathfrak{d} = \Gamma \cap \Gamma^{-1} = \mathcal{S}.$$

Note that for $y \leq_L \Gamma$, $y \in \mathfrak{d}$ if and only if $y \in \Gamma$. Hence (4) gives

$$\ker(p_{\mathfrak{d}}|_{L_{\leq \Gamma}}) = L_{< \Gamma}.$$

Its image therefore has basis $p_{\mathfrak{d}}(a_v) : v \in \Gamma$. For each $v \in \Gamma$, we have $a_v e = a_v$, so

$$p_{\mathfrak{d}}(a_v) = p_{\mathfrak{d}}(a_v)e \in A_{\mathfrak{d}}e.$$

Conversely, since $\mathbb{C}[W]e \subseteq L_{\leq \Gamma}$ and

$$p_{\mathfrak{d}}(\mathbb{C}[W]e) = p_{\mathfrak{d}}(\mathbb{C}[W])e = A_{\mathfrak{d}}e,$$

the image is all of $A_{\mathfrak{d}}e$. Thus $p_{\mathfrak{d}}$ induces the W -module isomorphism

$$[\Gamma] = L_{\leq \Gamma} / L_{< \Gamma} \simeq A_{\mathfrak{d}}e.$$

Similarly, $p_{\mathfrak{d}}(e\mathbb{C}[W]e) = eA_{\mathfrak{d}}e$. Recall that $e\mathbb{C}[W]e$ is spanned by the a_v for $v \in Q_I$, and every such v satisfies $v \leq_{LR} \mathfrak{d}$. By (4), the surviving classes are

$$\{p_{\mathfrak{d}}(a_v) : v \in Q_I \cap \mathfrak{d} = \mathcal{S}\},$$

which form a basis of $eA_{\mathfrak{d}}e$.

Finally, we pass from the a_v 's to D_v 's. Inverting this triangular change of basis (3) gives

$$D_v = a_v + \sum_{\substack{y \in Q_I \\ y < v}} n_{y,v} a_y \quad (v \in Q_I).$$

For $v \in \mathcal{S}$, projection kills precisely the terms in this sum whose indices are outside $\mathcal{S}$, leaving

$$p_{\mathfrak{d}}(D_v) = p_{\mathfrak{d}}(a_v) + \sum_{\substack{y \in \mathcal{S} \\ y < v}} n_{y,v} p_{\mathfrak{d}}(a_y).$$

This is again a triangular change of basis with diagonal entries 1. Hence the $p_{\mathfrak{d}}(D_v)$, $v \in \mathcal{S}$, form a basis of $eA_{\mathfrak{d}}e$, as required. $\square$

Lemma 3.2 (Multiplicity of the parabolic sign representation). *For every $R \in \mathcal{F}_{\mathfrak{d}}$,*

$$\dim(eR) = \dim \text{Hom}_W(R, [\Gamma]).$$

Proof. Lemma 3.1 gives $[\Gamma] \simeq A_{\mathfrak{d}}e$. The Wedderburn decomposition, as a left W -module, is

$$A_{\mathfrak{d}}e \simeq \bigoplus_{R \in \mathcal{F}_{\mathfrak{d}}} R \otimes (eR)^*.$$

Comparing multiplicities proves the formula. $\square$

Theorem 3.3 (Parabolic orbit comparison). *Let $I \subseteq S_0$, let w_I be the longest element of W_I , and let Γ be the left cell containing w_I . Suppose that*

$$[\Gamma] \simeq \bigoplus_{x \in X} R_x,$$

where X is a finite set and the R_x are pairwise nonisomorphic irreducible W -representations. Suppose that $x \mapsto v_x$ is a bijection

$$X \xrightarrow{\sim} \Gamma \cap \Gamma^{-1}.$$

For each $x \in X$, put $\mathcal{O}_x = \text{Supp}_H(R_x)$. Assume that the orbits $\mathcal{O}_x$ are pairwise distinct and that

$$\text{St}_H(v_x) \subseteq \overline{\mathcal{O}_x} \quad (x \in X).$$

Then, for every $x \in X$,

$$\text{St}_H(v_x) = \mathcal{O}_x, \quad R_x(D_{v_x}) \neq 0.$$

Proof. By Lemma 3.2, $\dim(eR_y) = 1$ for every $y \in X$ and $eR = 0$ for every other $R \in \mathcal{F}_\mathfrak{d}$. Thus

$$eA_\mathfrak{d}e \simeq \bigoplus_{y \in X} \text{End}_{\mathbb{C}}(eR_y) \simeq \mathbb{C}^X,$$

where an element of $eA_\mathfrak{d}e$ is sent to the scalars by which it acts on each one-dimensional space eR_y . Since $D_{v_x} = eD_{v_x}e$, it preserves each eR_y ; define $T = (T_{y,x})_{y,x \in X}$ by

$$R_y(D_{v_x})|_{eR_y} = T_{y,x} \text{id}_{eR_y}.$$

Since $R_y \in \mathcal{F}_\mathfrak{d}$, $R_y(D_{v_x}) = R_y(p_\mathfrak{d}(D_{v_x}))$ for all $x, y \in X$. Hence the x^{th} column of T is the list of scalars for $p_\mathfrak{d}(D_{v_x})$ in $\mathbb{C}^X$. Since $p_\mathfrak{d}(D_{v_x})$ form a basis of $eA_\mathfrak{d}e$ by Lemma 3.1, this implies that the columns of T form a basis of $\mathbb{C}^X$, so T is invertible.

If $T_{y,x} \neq 0$, then $R_y(D_{v_x}) \neq 0$. and Equation (6) give:

$$(7) \quad T_{y,x} \neq 0 \implies \mathcal{O}_y \subseteq \overline{\text{St}_H(v_x)} \subseteq \overline{\mathcal{O}_x}.$$

Order the distinct orbits by any total order extending closure inclusion, and use this order for both rows and columns. Then (7) implies that T is upper triangular. Since T is invertible, $T_{x,x} \neq 0$ and hence $R_x(D_{v_x}) \neq 0$. This gives

$$\mathcal{O}_x \subseteq \overline{\text{St}_H(v_x)} \subseteq \overline{\mathcal{O}_x}$$

and the result follows. $\square$

4. CLASSICAL PARTITIONS AND SPRINGER CORRESPONDENCES

Throughout Sections 4–8, let X and Y denote the types of $G^\vee$ and G , respectively, and put $\varepsilon = 1$ if $X = B$ and $\varepsilon = 0$ if $X = C$.

4.1. Partitions, special orbits, and good parity. Let $\lambda = [\lambda_1 \geq \lambda_2 \geq \dots]$ be a partition of an integer $m \geq 0$. Write $|\lambda|$ for its size, $\ell(\lambda)$ for its number of positive parts, λ^t for its transpose.

Nilpotent orbits in $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are identified with their Jordan partitions. In type B , such partitions have size $2n + 1$, and every even part has even multiplicity. In type C , they have size $2n$, and every odd part has even multiplicity.

Definition 4.1. A nilpotent orbit of type B or C has **good parity** if all parts of its row partition are odd in type B or even in type C .

Following [21, §2.1], we write a nilpotent orbit $\mathcal{O}^\vee$ in $\mathfrak{g}^\vee$ as

$$\mathcal{O}^\vee = [r_1 \geq r_2 \geq \dots \geq r_\ell]$$

in terms of its row lengths. For a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, we use the column lengths of its partition:

$$\mathcal{O} = (c_1 \geq c_2 \geq \dots \geq c_\ell) = [c_1 \geq c_2 \geq \dots \geq c_\ell]^t.$$

When needed, we may add zero entries at the end.

For partitions $\lambda = [\lambda_1 \geq \dots \geq \lambda_\ell]$ and $\mu = [\mu_1 \geq \dots \geq \mu_\ell]$ of m , with zeros adjoined so that they have the same length,

$$\lambda \preceq \mu \iff \sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i \quad \text{for every integer } k \geq 1,$$

which also gives closure order on nilpotent orbits. For $X \in \{B, C\}$ and a partition p of the appropriate size, its **X -collapse** p_X is the greatest partition of type X dominated by p [8, §6.3]. The partition p^+ is obtained by increasing its largest part by one, with $\emptyset^+ = [1]$. For a nonempty partition p , p^- is obtained by decreasing its smallest positive part by one and deleting a resulting zero.

4.2. Bipartitions and symbols. We now study Weyl group representations of G and $G^\vee$. Let $W_m = (\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m$. For a partition λ of m , inflate the Specht representation S^λ of S_m to W_m through $W_m \twoheadrightarrow S_m$, and let κ_m be the coordinate-sign character

$$\kappa_m : (\epsilon_1, \dots, \epsilon_m; \pi) \mapsto \prod_{i=1}^m \epsilon_i.$$

Every irreducible representation of $W = W_n$ is of the form

$$E(\lambda; \mu) = \text{Ind}_{W_a \times W_b}^{W_n} (S^\lambda \boxtimes (S^\mu \otimes \kappa_b)),$$

where $(\lambda; \mu)$ is a bipartition of n , that is, $|\lambda| = a$, $|\mu| = b$, and $a + b = n$. In particular,

$$E((n); \emptyset) = \mathbf{1}, \quad E(\emptyset; 1^n) = \text{sgn},$$

where 1^n is a column of length n . Tensoring with sign gives

$$(8) \quad E(\lambda; \mu) \otimes \text{sgn} = E(\mu^t; \lambda^t).$$

Definition 4.2. Let E be an irreducible Weyl-group representation. Write $a(E)$ for the least exponent in its generic degree for the equal-parameter Hecke algebra, and $b(E)$ for its least degree of occurrence in the symmetric algebra of the reflection representation. The representation is **special** (in the sense of Lusztig [13]) if

$$a(E) = b(E).$$

A nilpotent orbit $\mathcal{O}$ is **special** if its Springer representation $\text{Spr}(\mathcal{O}, \mathbf{1})$ is special.

For $X \in \{B, C\}$, a type- X partition λ is special if and only if λ^t is also of type X [11, §1.3.1]. In particular, every good-parity orbit is special.

Definition 4.3 (Partition correspondence). Let $\varepsilon \in \{0, 1\}$ and let p be a partition of $2n + \varepsilon$. If p is of even length, adjoin a zero so that p has odd length $2m + 1$. Its **beta-numbers** are the distinct integers

$$p_j + 2m + 1 - j \quad (1 \leq j \leq 2m + 1).$$

Define $\mathcal{P}_\varepsilon(n)$ to be the set of such partitions for which exactly $m + 1$ beta-numbers are congruent to ε modulo two and exactly m have the other parity.

Let $X = B$ when $\varepsilon = 1$ and $X = C$ when $\varepsilon = 0$. For $p \in \mathcal{P}_\varepsilon(n)$, separate its beta-numbers by parity, listing each parity in decreasing order. Define the bipartition $(\lambda; \mu)$ by requiring the i th beta-number in each list to be

$$\begin{aligned} 2(\lambda_i + m + 1 - i) + \varepsilon & \quad (1 \leq i \leq m + 1, \text{ parity } \varepsilon), \\ 2(\mu_i + m - i) + 1 - \varepsilon & \quad (1 \leq i \leq m, \text{ parity } 1 - \varepsilon), \end{aligned}$$

and set

$$\Pi_X(p) = E(\lambda; \mu)$$

the irreducible representation of W corresponding to p .

The map Π_X is the injective partition correspondence of [17, §2, pp. 79–80]. It is independent of the number of zero parts adjoined and satisfies

$$\Pi_B([2n + 1]) = \Pi_C([2n]) = \mathbf{1}.$$

Both sets $\mathcal{P}_\varepsilon(n)$ are stable under transposition [17, Lemma 2.10].

Definition 4.4. Let $E(\lambda; \mu)$ be an irreducible representation of W . Its **Lusztig symbol** is defined by

$$\begin{pmatrix} \lambda_1 + m & \cdots & \lambda_m + 1 & \lambda_{m+1} \\ \mu_1 + m - 1 & \cdots & \mu_m & \end{pmatrix}.$$

Both rows are strictly decreasing and nonnegative.

An entry in the Lusztig symbol is **isolated** if it occurs only once in the two rows. Also, two Lusztig symbols of lengths $m + 1$ and m are **congruent** if their combined entry multisets agree after shifts to the same row lengths. A symbol is **special** if its entries interleave, equivalently if

$$\lambda_i + 1 \geq \mu_i \geq \lambda_{i+1} \quad (1 \leq i \leq m).$$

A **family** consists of the representations whose Lusztig symbols belong to a fixed congruence class. Each congruence class has a unique special symbol.

Proposition 4.5 (Classical partition and Springer correspondence). *For $X \in \{B, C\}$, every nilpotent-orbit partition of type X belongs to $\mathcal{P}_\varepsilon(n)$, and*

$$\Pi_X(p) = \text{Spr}_X(p, \mathbf{1}).$$

Here Spr_B and Spr_C denote ordinary Springer correspondence for SO_{2n+1} and Sp_{2n} , respectively. For every $p \in \mathcal{P}_\varepsilon(n)$,

$$(9) \quad \Pi_X(p^t) = \Pi_X(p) \otimes \text{sgn}.$$

For special orbits in Langlands-dual Lie algebras,

$$(10) \quad \text{Spr}_G(\mathcal{O}, \mathbf{1}) \otimes \text{sgn} = \text{Spr}_{G^\vee}(d_{BV}(\mathcal{O}), \mathbf{1}).$$

Proof. The first assertion is the classical partition construction of ordinary Springer correspondence, in precisely the normalization of [17, §2, pp. 79–80]. Equation (9) is [17, Lemma 2.10]. Equation (10) is the sign characterization of Barbasch–Vogan duality on special orbits [6]. $\square$

Definition 4.6. The type- X u -symbol of $E(\lambda; \mu)$ is

$$(11) \quad \begin{pmatrix} \lambda_1 + 2m & \cdots & \lambda_m + 2 & \lambda_{m+1} \\ \mu_1 + 2m - 1 - \varepsilon & \cdots & \mu_m + 1 - \varepsilon & \end{pmatrix}.$$

Two u -symbols of lengths $m+1$ and m are **similar** if their combined entry multisets agree after shifts to the same row lengths. A u -symbol is **distinguished** if its rows, written increasingly as

$$\begin{pmatrix} a_0 & \cdots & a_{m-1} & a_m \\ b_0 & \cdots & b_{m-1} & \end{pmatrix},$$

satisfy $a_0 \leq b_0 \leq a_1 \leq \cdots \leq b_{m-1} \leq a_m$.

Proposition 4.7 (Ordinary Springer support and u -symbols). *Two irreducible W -representations have the same ordinary Springer support for a fixed type $X \in \{B, C\}$ if and only if their type- X u -symbols are similar. Each similarity class has a unique distinguished representative, up to shift, and this representative corresponds to the trivial local system on its orbit.*

Proof. These assertions are the classical u -symbol description of ordinary Springer correspondence; see [3, §§3.2, 3.4, 4.1–4.2] and [11, Appendix A.2]. $\square$

4.3. Canonical quotients and family labels. For $\xi \in \mathcal{O}$, let $A(\mathcal{O}) = Z_G(\xi)/Z_G(\xi)^\circ$, and let $\overline{A}(\mathcal{O})$ be its Lusztig canonical quotient, which is an elementary abelian 2-group in our cases. For a finite abelian group A , write $\widehat{A} = \text{Hom}(A, \mathbb{C}^\times)$.

Let $\mathcal{F}_\mathcal{O}$ be the family containing $\text{Spr}_G(\mathcal{O}, \mathbf{1})$. The Springer-compatible family parametrization is

$$\kappa_\mathcal{O} : \mathcal{F}_\mathcal{O} \longrightarrow \overline{A}(\mathcal{O}) \times \widehat{\overline{A}(\mathcal{O})}.$$

It is an injection [3, §§3.1–3.4]. On its image we define $\sigma_\mathcal{O}(x, \chi) = \kappa_\mathcal{O}^{-1}(x, \chi)$; in particular, $\sigma_\mathcal{O}(0, \mathbf{1}) = \text{Spr}_G(\mathcal{O}, \mathbf{1})$. The **special piece** of $\mathcal{O}$ is

$$(12) \quad \mathcal{P}(\mathcal{O}) = \overline{\mathcal{O}} \setminus \bigcup_{\substack{\mathcal{O}' \text{ special} \\ \overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}}}} \overline{\mathcal{O}'}.$$

The generators used for the family labels are specified in Section 5.

A left cell $\mathcal{C}$ is **pure for $\mathcal{O}$** if all pairs $(x, \mathbf{1})$ occur and

$$[\mathcal{C}] = \bigoplus_{x \in \overline{A}(\mathcal{O})} \sigma_\mathcal{O}(x, \mathbf{1}).$$

5. PAIRED PARTITIONS AND CANONICAL CLASS LABELS

5.1. The good-parity orbit and its dual. Let $\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ have good parity, with X , Y , and ε as in Section 4. In the notations of [21, Proposition 2.2]:

$$(13) \quad \mathcal{O}^\vee = [2h_1 + \varepsilon, \dots, 2h_{2z+1} + \varepsilon] = [r_1 \geq \dots \geq r_{2f+1}] \cup \bigcup_{j=1}^y [\beta_j, \beta_j].$$

Here $z = f + y$ and $h_1 \geq \dots \geq h_{2z+1} \geq 0$ are integers. When $X = C$, we adjoin a final zero row if necessary. The pairs $[\beta_j, \beta_j]$ are precisely the equal pairs among the designated pairs $[2h_{2k-1} + \varepsilon, 2h_{2k} + \varepsilon]$ for $X = B$, or $[2h_{2k} + \varepsilon, 2h_{2k+1} + \varepsilon]$ for $X = C$. Put

$$p_i = 2i - \varepsilon \quad (1 \leq i \leq f).$$

We pair up the ‘retained’ rows r_i by $[r_{p_i}, r_{p_i+1}]$, with $r_{p_i} > r_{p_i+1}$. In particular, r_{2f+1} for $X = B$ and r_1 for $X = C$ remains unpaired.

The classical duality formula [21, Lemma 4.2 and §4.2] gives the following column expression:

$$(14) \quad \mathcal{O} := d_{BV}(\mathcal{O}^\vee) = \bigcup_{i=1}^f (r_{p_i} - 1, r_{p_i+1} + 1) \cup \bigcup_{j=1}^y (\beta_j, \beta_j) \cup \begin{cases} (r_{2f+1} - 1), & \varepsilon = 1, \\ (r_1 + 1, 0), & \varepsilon = 0. \end{cases}$$

If r_{p_i} is the ℓ -th row length of $\mathcal{O}^\vee$, then r_{p_i+1} is its $(\ell + 1)$ -st row length, and the ℓ -th and $(\ell + 1)$ -st column lengths of $\mathcal{O}$ are $r_{p_i} - 1$ and $r_{p_i+1} + 1$, respectively.

Set $\mathcal{R} = \{1, \dots, f\}$. The orbits $\mathcal{O}^\vee$ and $\mathcal{O}$ will remain fixed throughout Sections 5–8.

5.2. The partitions indexed by subsets of $\mathcal{R}$.

Definition 5.1 (The orbits $\mathcal{O}_J^\vee$). For every $J \subseteq \mathcal{R}$, define the row partition

$$(15) \quad \tilde{\rho}_J = \bigcup_{i \in J} [r_{p_i} - 1, r_{p_i+1} + 1] \cup \bigcup_{i \in \mathcal{R} \setminus J} [r_{p_i}, r_{p_i+1}] \cup \bigcup_{j=1}^y [\beta_j, \beta_j] \cup \begin{cases} [r_{2f+1}], & \varepsilon = 1, \\ [r_1], & \varepsilon = 0. \end{cases}$$

Set

$$(16) \quad \mathcal{O}_J^\vee := (\tilde{\rho}_J)_X.$$

Note that only the selected pairs in (15) have parity $1 - \varepsilon$. The collapse rule of [8, §6.3] therefore gives

$$(17) \quad [r_{p_i} - 1, r_{p_i+1} + 1] \mapsto \begin{cases} [r_{p_i} - 1, r_{p_i+1} + 1], & r_{p_i} - r_{p_i+1} = 2, \\ [r_{p_i} - 2, r_{p_i+1} + 2], & r_{p_i} - r_{p_i+1} \geq 4 \end{cases} \quad (i \in J).$$

The unselected pairs, all pairs $[\beta_j, \beta_j]$, and the unpaired row remain unchanged. In particular, $\tilde{\rho}_\emptyset = \mathcal{O}^\vee$ and $\mathcal{O}_\emptyset^\vee = \mathcal{O}^\vee$.

Definition 5.2 (The orbits $\mathcal{O}_J$). For every $J \subseteq \mathcal{R}$, define $\mathcal{O}_J$ by the following column expression:

$$(18) \quad \mathcal{O}_J = \bigcup_{i \in J} (r_{p_i}, r_{p_i+1}) \cup \bigcup_{i \in \mathcal{R} \setminus J} (r_{p_i} - 1, r_{p_i+1} + 1) \cup \bigcup_{j=1}^y (\beta_j, \beta_j) \cup \begin{cases} (r_{2f+1} - 1), & \varepsilon = 1, \\ (r_1 + 1, 0), & \varepsilon = 0. \end{cases}$$

Thus the selected column pairs of $\mathcal{O}$ are replaced by the corresponding pairs of lengths in $\mathcal{O}^\vee$, and every other column is unchanged. In particular, $\mathcal{O}_\emptyset = \mathcal{O}$. By [21, Proposition 2.3], this is a nilpotent orbit of type Y .

We now construct a bipartition $(\lambda^J; \mu^J)$ for each $J \subseteq \mathcal{R}$. Start with the list $h_1, \dots, h_{2z+1}$ in (13). Define $(\lambda^\emptyset; \mu^\emptyset)$ by:

$$\lambda_j^\emptyset = h_{2j-1} \quad (1 \leq j \leq z+1), \quad \mu_j^\emptyset = h_{2j} + \varepsilon \quad (1 \leq j \leq z).$$

For $i \in J$ such that row pairs $[2h_k + \varepsilon, 2h_{k+1} + \varepsilon] = [r_{p_i}, r_{p_i+1}]$ match. Then there exists a unique j such that $k = 2j - \varepsilon$ and $k + 1 = 2j + 1 - \varepsilon$, and we let

$$(19) \quad (\lambda_{j+1-\varepsilon}^J, \mu_j^J) = (\mu_j^\emptyset - \varepsilon, \lambda_{j+1-\varepsilon}^\emptyset + \varepsilon).$$

After applying the changes (19) for all $i \in J$, define

$$R_J := E(\lambda^J; \mu^J).$$

Lemma 5.3. *For every $J \subseteq \mathcal{R}$, $\tilde{\rho}_J \in \mathcal{P}_\varepsilon(n)$, and*

$$(20) \quad \Pi_Y(\mathcal{O}_J^t) = \Pi_X(\tilde{\rho}_J) = R_J.$$

All R_J have congruent symbols. The symbol of $R_\emptyset$ is special and has exactly $2f + 1$ isolated entries.

Proof. We only present the proof for $\tilde{\rho}_J$. The changes $[r_{p_i}, r_{p_i+1}] \mapsto [r_{p_i} - 1, r_{p_i+1} + 1]$ for each $i \in J$ exchange the parities of the corresponding beta-numbers. By directly applying the algorithm in Definition 4.3, it gives the $(\lambda^J; \mu^J)$ as described before the lemma.

The symbol of $R_\emptyset$ has weakly decreasing interleaved entries

$$h_k + \left\lfloor \frac{2z + 1 - k + \varepsilon}{2} \right\rfloor \quad (1 \leq k \leq 2z + 1).$$

and hence it is special. The β -pairs give repeated entries, while the pairs $[r_{p_i}, r_{p_i+1}]$ for $i \in \mathcal{R}$ and the unpaired row give exactly $2f + 1$ isolated entries. Applying (19) for all $i \in J$ and the formula for computing Lusztig symbols in Definition 4.4, it exchanges its two Lusztig symbol entries between the rows, so all R_J have congruent symbols. $\square$

5.3. The generators and the family labels. We now describe $\overline{A}(\mathcal{O})$ for

$$\mathcal{O} = (c_1 \geq c_2 \geq \dots) = \bigcup_{i=1}^f (r_{p_i} - 1, r_{p_i+1} + 1) \cup \bigcup_{j=1}^y (\beta_j, \beta_j) \cup \begin{cases} (r_{2f+1} - 1), & \varepsilon = 1, \\ (r_1 + 1, 0), & \varepsilon = 0. \end{cases}$$

as in (14) using the classification in [11, §1.2.2 and Proposition 1.3]. Under their notations, the multiplicity and height of a certain row length q are given by:

$$m_\lambda(q) = c_q - c_{q+1}, \quad h_\lambda(q) = c_q.$$

For $\xi \in \mathcal{O}$ and each position q such that $c_q > c_{q+1}$ and $q \equiv 1 - \varepsilon \pmod{2}$, let x_q denote the component of determinant -1 in the orthogonal factor $O_{c_q - c_{q+1}}$ of the centralizer of ξ in the full isometry group, with the identity component in every other factor. For $\varepsilon = 1$, they generate $A(\mathcal{O})$; for $\varepsilon = 0$, only products with an even number of x_q 's generate $A(\mathcal{O})$.

Now study $\overline{A}(\mathcal{O})$. For $i \in \mathcal{R}$, let q_i be the column number such that

$$(c_{q_i-1}, c_{q_i}) = (r_{p_i} - 1, r_{p_i+1} + 1), \quad q_i \equiv 1 - \varepsilon \pmod{2}.$$

Put $x_0 = 1$, $q_0 = 0$ when $\varepsilon = 1$, and $q_0 = 1$ when $\varepsilon = 0$. Then for $i \in \mathcal{R}$ and $J \subseteq \mathcal{R}$, define

$$(21) \quad \bar{\theta}_i = \overline{x_{q_i-1} x_{q_i}}, \quad x_J = \sum_{i \in J} \bar{\theta}_i = \overline{\prod_{i \in J} x_{q_i-1} x_{q_i}}.$$

For $f > 0$ and $\varepsilon = 1$, the first generator is $\bar{\theta}_1 = \overline{x_{q_1}}$; for $\varepsilon = 0$, every displayed product has an even number of factors.

We claim that the $\bar{\theta}_i$'s generate $\overline{A}(\mathcal{O})$ in [11, Proposition 1.3]. Indeed, one has

$$c_{q_i} > c_{q_i+1}, \quad c_{q_i-2} > c_{q_i-1}.$$

Since $c_{q_i} = r_{p_i+1} + 1$ has parity $1 - \varepsilon$, and the β -columns have the opposite parity [11, Equation (1), Proposition 1.3] implies that the classes $\overline{x_{q_i-2} x_{q_i}}$ are precisely the basis elements of $\overline{A}(\mathcal{O}) \cong (\mathbb{Z}/2\mathbb{Z})^f$. In particular, $J \mapsto x_J$ is a bijection between subsets of $\mathcal{R}$ and $\overline{A}(\mathcal{O})$.

Remark 5.4. The above construction of $\overline{A}(\mathcal{O})$ is compatible with that of [21, §2.2], where the author used column sizes c_q instead of column number q to denote the elements $\bar{\theta}_i \in \overline{A}(\mathcal{O})$.

Proposition 5.5 (Canonical classes and family labels). *The orbits in $\mathcal{P}(\mathcal{O})$ are exactly the $\mathcal{O}_J$, for $J \subseteq \mathcal{R}$. For every such J ,*

$$(22) \quad \sigma_{\mathcal{O}}(x_J, \mathbf{1}) = \text{Spr}_G(\mathcal{O}_J, \mathbf{1}).$$

Consequently every pair $(x, \mathbf{1})$ with $x \in \overline{A}(\mathcal{O})$ occurs, and

$$(23) \quad R_J = \sigma_{\mathcal{O}}(x_J, \mathbf{1}) \otimes \text{sgn} = \Pi_X(\tilde{\rho}_J).$$

Proof. By Lemma 5.3 and (9), it suffices to show that

$$\sigma_{\mathcal{O}}(x_J, \mathbf{1}) = \text{Spr}_G(\mathcal{O}_J, \mathbf{1}).$$

The elementary collapses defining $\mathcal{O}_J$ give all the orbits in $\mathcal{P}(\mathcal{O})$; see [14, Propositions 1.8 and 2.8, §6.1]. By [14, Theorems 0.2 and 0.4], their trivial-local-system Springer representations belong to $\mathcal{F}_{\mathcal{O}}$ and have family labels of the form $(x, \mathbf{1})$. It remains to identify x in our notation.

Fix $i \in \mathcal{R}$ and put $q = q_i$, $u = c_{q-1} = r_{p_i} - 1$, $v = c_q = r_{p_i+1} + 1$. The column change $(u, v) \mapsto (u+1, v-1)$ is the row collapse

$$(24) \quad [q, (q-1)^{u-v}, q-2] \mapsto [(q-1)^{u-v+2}].$$

Here $u-v$ is even, and for $q = 2$ the last entry on the left is an adjoined zero. This is the modification given in [14, Propositions 1.8 and 2.8] and [11, §1.3.1, formulas (2)–(3)]. Under the Springer-compatible identification with $\bar{A}(\mathcal{O})$, this endpoint pair is

$$(25) \quad \overline{x_q x_{q-2}} = \overline{x_{q-1} x_{q_i}} = \bar{\theta}_i.$$

by [11, Theorem A.5 and §A.7]. Therefore, the element $x_J = \sum_{i \in J} \bar{\theta}_i$ correspond to the collapsed orbit $\mathcal{O}_J \in \mathcal{P}(\mathcal{O})$, and [14, Theorem 0.4] gives

$$\kappa_{\mathcal{O}}(\text{Spr}_G(\mathcal{O}_J, \mathbf{1})) = (x_J, \mathbf{1}).$$

This proves (22). Finally, Lemma 5.3 and (9) give (23). $\square$

Example 5.6. Let $G^{\vee} = \text{Sp}_{18}(\mathbb{C})$ and $\mathcal{O}^{\vee} = [8, 6, 2, 2]$. The row decomposition (13) is

$$[r_1, r_2, r_3, r_4, r_5] = [8, 6, 2, 2, 0], \quad f = 2, \quad y = 0.$$

The equal rows of length 2 belong to different designated pairs, namely $[6, 2]$ and $[2, 0]$, and therefore do not form a β -pair. Formula (14) gives

$$\mathcal{O} = (9, 5, 3, 1, 1, 0), \quad \mathcal{R} = \{1, 2\}.$$

The corresponding column pairs are $(5, 3)$ and $(1, 1)$, so $q_1 = 3$, $q_2 = 5$ (with $q_0 = 1$). Hence

$$\bar{\theta}_1 = \overline{x_1 x_3}, \quad \bar{\theta}_2 = \overline{x_3 x_5}.$$

The special Lusztig symbol of $\mathcal{O}$, with rows written increasingly, is

$$\begin{pmatrix} 0 & 1 & 3 & 4 & 6 \\ 1 & 2 & 3 & 5 & \end{pmatrix}.$$

Its isolated entries are 0, 2, 4, 5, 6. The changes at $i = 1$ and $i = 2$ transfer the pairs $\{2, 4\}$ and $\{5, 6\}$, respectively. Proposition 5.5 assigns the labels $(0, \mathbf{1})$, $(\bar{\theta}_1, \mathbf{1})$, $(\bar{\theta}_2, \mathbf{1})$, and $(\bar{\theta}_1 + \bar{\theta}_2, \mathbf{1})$ to the trivial Springer representations of $\mathcal{O}_J$ for $J = \emptyset, \{1\}, \{2\}, \{1, 2\}$, respectively.

6. EVALUATION OF GENERALIZED DUALITY

Retain the notation of Section 5.

Proposition 6.1 (Evaluation of generalized duality). *For every $J \subseteq \mathcal{R}$,*

$$(26) \quad d_S(\mathcal{O}, x_J) = \mathcal{O}_J^{\vee}.$$

Its row partition is obtained from (15) by applying the collapse rule (17). The orbits $\mathcal{O}_J^{\vee}$ are pairwise distinct as J varies.

Proof. Retain the column lengths c_k and the positions q_i used in (21). Fix $J \subseteq \mathcal{R}$, and let $\ell_1 > \dots > \ell_s > 0$ be the positive integers that occur an odd number of times among $(q_i - 2, q_i)$ for $i \in J$. By (21), one has

$$x_J = \prod_{i \in J} \overline{x_{q_i-1} x_{q_i}} = \prod_{i \in J} \overline{x_{q_i-2} x_{q_i}} = \overline{x_{\ell_1} \cdots x_{\ell_s}}.$$

In particular, each ℓ_j is a column position satisfying $c_{\ell_j} > c_{\ell_j+1}$ and $\ell_j \equiv 1 - \varepsilon \pmod{2}$. Thus $\mathcal{O}$ has a Jordan block of each size ℓ_j . If $\varepsilon = 0$, no endpoint is zero, so s is even.

Choose $\xi \in \mathcal{O}$ and let V_- be a nondegenerate orthogonal sum of one Jordan block of each size ℓ_j in the natural representation V . Put $V_+ = V_-^{\perp}$. The involution $s_J = -1_{V_-} \oplus 1_{V_+}$

belongs to $Z_G(\xi)$, and its image in $\overline{A}(\mathcal{O})$ is x_J . Writing $\dim V_- = 2m$, the pseudo-Levi subgroup $L_J = Z_G(s_J)^\circ$ of G has type

$$\begin{cases} C_m \times C_{n-m}, & \varepsilon = 1, \\ D_m \times B_{n-m}, & \varepsilon = 0, \end{cases}$$

with factors of rank zero omitted. The Jordan partitions on V_- and V_+ are $\lambda = [\ell_1, \dots, \ell_s]$ and $\eta = \mathcal{O} \setminus \lambda$, respectively, where the difference removes the indicated rows.

Sommers' duality is defined using Lusztig–Spaltenstein duality in L_J and truncated induction of the resulting Springer representation with trivial local system. The classical formula of [18, Theorem 12] gives

$$(27) \quad d_S(\mathcal{O}, x_J) = (\lambda^t + d_{BV}(\eta))_X,$$

where $+$ denotes componentwise addition of row partitions, with zeros adjoined when their lengths differ. Note that the partition η need not be special.

Write the column partition of λ as $\lambda^t = [a_1 \geq a_2 \geq \dots]$, where $a_k = \#\{j : \ell_j \geq k\}$. Cancelling repeated x_q 's changes each cumulative count by an even number. Thus, for every $k \geq 1$,

$$a_k \equiv \sum_{i \in J} (\mathbf{1}_{k \leq q_i - 2} + \mathbf{1}_{k \leq q_i}) \equiv \sum_{i \in J} \mathbf{1}_{q_i - 1 \leq k \leq q_i} \pmod{2},$$

where $\mathbf{1}_P$ is 1 if P holds and 0 otherwise. The pairs of positions $\{q_i - 1, q_i\}$ are disjoint, so a_k is odd precisely at $k = q_i - 1, q_i$ for $i \in J$. Since every ℓ_j has parity $1 - \varepsilon$ and the ℓ_j are distinct, we also have $a_{q_i - 1} = a_{q_i}$ and $a_k - a_{k+1} \in \{0, 1\}$. Hence

$$\eta = (e_1 \geq e_2 \geq \dots), \quad e_k = c_k - a_k.$$

By [8, §6.3],

$$d_{BV}(\eta) = \begin{cases} ((\eta^t)^+)_B, & \varepsilon = 1, \\ ((\eta^t)^-)_C, & \varepsilon = 0 \end{cases}$$

For $i \in \mathcal{R}$, the entries $e_{q_i - 1}, e_{q_i}$ have parity ε if $i \in J$ and parity $1 - \varepsilon$ otherwise. At the positions of (β_j, β_j) their parity is ε . Also, e_{2z+1} is even if $\varepsilon = 1$, and e_1 is odd if $\varepsilon = 0$. For $1 \leq k \leq 2z$ with $k \equiv 1 - \varepsilon \pmod{2}$, we have

$$(28) \quad e_k > e_{k+1}$$

whenever either entry has parity $1 - \varepsilon$. If e_k and e_{k+1} have different parities, then $e_k > e_{k+1}$ because the entries are weakly decreasing. Otherwise both have parity $1 - \varepsilon$. By the parities of the columns in (14), the integers a_k and a_{k+1} are then even. Since $a_k - a_{k+1} \in \{0, 1\}$, they are equal. Thus $e_k - e_{k+1} = c_k - c_{k+1} \geq 2$, as required.

In the collapse rule [8, §6.3], list the entries of η^t of parity $1 - \varepsilon$ in decreasing order, including the final zero if $\varepsilon = 1$ and $e_{2z+1} = 0$. These entries occur in pairs at the unselected retained positions, together with the endpoint required by the $+$ or $-$ operation. Increasing the first entry when $\varepsilon = 1$, or decreasing the last positive entry when $\varepsilon = 0$, shifts the pairing across these retained pairs. If $\varepsilon = 0$ and the smallest positive entry is even, its preliminary decrement is undone by the subsequent C -collapse, which decreases the preceding odd entry instead. Thus the upper entry of each unselected pair is the second entry of one collapse pair, and its lower entry is the first entry of the next. Entries at selected pairs and at β -pairs are unchanged. The strict inequalities above show that the collapse pairs are unequal, so each $[u, v]$ becomes $[u - 1, v + 1]$. Consequently, by (28), write

$$d_{BV}(\eta) = [f_1, f_2, \dots].$$

Then $d_{BV}(\eta)$ is obtained as follows:

$$[f_{q_i - 1}, f_{q_i}] = \begin{cases} [e_{q_i - 1}, e_{q_i}], & i \in J, \\ [e_{q_i - 1} + 1, e_{q_i} - 1], & i \in \mathcal{R} \setminus J. \end{cases}$$

At the positions of (β_j, β_j) , we have $f_k = e_k$, while

$$f_{2z+1} = e_{2z+1} + 1 \quad (\varepsilon = 1), \quad f_1 = e_1 - 1 \quad (\varepsilon = 0).$$

Since $a_k + e_k = c_k$, it follows that

$$\lambda^t + d_{BV}(\eta) = [a_1 + f_1, \dots, a_{2z+1} + f_{2z+1}] = \tilde{\rho}_J.$$

Equation (27) now gives $d_S(\mathcal{O}, x_J) = (\tilde{\rho}_J)_X = \mathcal{O}_J^\vee$. The row formula follows from (17). Each selected pair changes its upper row, whereas each unselected pair leaves that row unchanged. Since the changes preserve the positions of the pairs, the resulting partition determines J . Thus the orbits $\mathcal{O}_J^\vee$ are pairwise distinct. $\square$

Example 6.2. Let $n = 7$ and $\mathcal{O}^\vee = [r_1, r_2, r_3, r_4, r_5] = [7, 3, 3, 1, 1]$, i.e. $f = 2$, and $y = 0$. Formula (14) gives $\mathcal{O} = (6, 4, 2, 2, 0)$, with retained column pairs $(6, 4)$ and $(2, 2)$. Here $q_1 = 2$, $q_2 = 4$, and

$$\mathcal{R} = \{1, 2\}, \quad \bar{\theta}_1 = \overline{x_2}, \quad \bar{\theta}_2 = \overline{x_2 x_4}.$$

Propositions 5.5 and 6.1 give

J	x_J	$\mathcal{O}_J$	$\mathcal{O}_J^\vee$
$\emptyset$	0	$(6, 4, 2, 2, 0)$	$[7, 3, 3, 1, 1]$
$\{1\}$	$\bar{\theta}_1$	$(7, 3, 2, 2, 0)$	$[5, 5, 3, 1, 1]$
$\{2\}$	$\bar{\theta}_2$	$(6, 4, 3, 1, 0)$	$[7, 3, 2, 2, 1]$
$\{1, 2\}$	$\bar{\theta}_1 + \bar{\theta}_2$	$(7, 3, 3, 1, 0)$	$[5, 5, 2, 2, 1]$

The corresponding bipartitions of R_J , in this order, are

$$((3, 1); (2, 1)), \quad ((1, 1); (4, 1)), \quad ((3); (2, 2)), \quad ((1); (4, 2)).$$

Selection of $i = 2$ retains the equal row pair $[2, 2]$ under collapse. Selection of $i = 1$ changes $[6, 4]$ into $[5, 5]$; for instance $\tilde{\rho}_{\{1\}} = [6, 4, 3, 1, 1]$.

7. ORDINARY SPRINGER SUPPORTS

For $J \subseteq \mathcal{R}$, Proposition 5.5 gives $R_J = \sigma_{\mathcal{O}}(x_J, \mathbf{1}) \otimes \text{sgn}$. We shall show that its ordinary Springer support is $\mathcal{O}_J^\vee$. The local system on this orbit need not be trivial, even though the character in the family label $(x_J, \mathbf{1})$ is trivial.

Proposition 7.1. *For every $J \subseteq \mathcal{R}$, the representations $R_J = \Pi_X(\tilde{\rho}_J)$ and $\Pi_X(\mathcal{O}_J^\vee) = \text{Spr}_{G^\vee}(\mathcal{O}_J^\vee, \mathbf{1})$ have similar type- X u -symbols. Consequently*

$$(29) \quad \text{Supp}_{G^\vee}(R_J) = \mathcal{O}_J^\vee = (\tilde{\rho}_J)_X.$$

Moreover, $R_J = \text{Spr}_{G^\vee}(\mathcal{O}_J^\vee, \mathbf{1})$ if and only if $r_{p_i} - r_{p_i+1} = 2$ for every $i \in J$.

Proof. We show that R_J and $\Pi_X(\mathcal{O}_J^\vee)$ have similar u -symbols, and hence the same ordinary Springer support. By definition, this means that their combined entry multisets agree. We compare these multisets pair by pair, using the common odd length $2z + 1$ for the partitions.

By (17), the unselected pairs, the β -pairs, and the unpaired row are unchanged; so are selected pairs with original gap two. Thus only selected pairs with original gap at least four can contribute different u -symbol entries.

Consider such a pair in rows $k, k + 1$ of $\mathcal{O}^\vee$, where $k = 2j - \varepsilon$ and $h_k - h_{k+1} \geq 2$. The row changes in (17) stay within the pair's original interval of row lengths, so its positions are preserved. The pair contributes one beta-number of each parity in both partitions; all entries outside the changed pairs therefore keep their indices and values in the two u -symbol rows. By (11) and (17), this pair contributes the same unordered pair

$$\{h_k + 2z - k, h_{k+1} + 2z + 1 - k\}$$

to both u -symbols. The two entries lie in opposite rows and are exchanged when passing from R_J to $\Pi_X(\mathcal{O}_J^\vee)$, in either type B or type C . Thus every changed pair preserves the combined multiset. Together with the unchanged entries, this proves similarity. Propositions 4.7 and 4.5 now give

$$\text{Supp}_{G^\vee}(R_J) = \text{Supp}_{G^\vee}(\Pi_X(\mathcal{O}_J^\vee)) = \mathcal{O}_J^\vee,$$

which is (29).

Finally, the local system of R_J on this common support is trivial if and only if $R_J = \Pi_X(\mathcal{O}_J^\vee)$. Since $R_J = \Pi_X(\tilde{\rho}_J)$ and Π_X is injective, this is equivalent to $\tilde{\rho}_J = \mathcal{O}_J^\vee$. By (17), that equality holds exactly when every selected pair has original gap two. $\square$

Example 7.2. Continue Example 6.2 with $J = \{1\}$. Proposition 7.1 gives

$$R_J = E((1, 1); (4, 1)), \quad \text{Supp}_{G^\vee}(R_J) = \mathcal{O}_J^\vee = [5, 5, 3, 1, 1].$$

The u -symbol of R_J and the distinguished u -symbol with the same combined multiset are, with increasing rows,

$$\begin{pmatrix} 0 & 3 & 5 \\ 1 & 6 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 6 \\ 1 & 5 & \end{pmatrix},$$

respectively. The second array represents $E((2, 1); (3, 1)) = \text{Spr}_{G^\vee}(\mathcal{O}_J^\vee, \mathbf{1})$. Since these representations are distinct, the ordinary Springer pair of R_J consists of $[5, 5, 3, 1, 1]$ and a nontrivial local system.

8. THE CANONICAL CELL

8.1. Insertion partitions and the natural representation. We use insertion partitions to label the constituents of a left-cell representation and to bound the Jordan type of a Springer–Steinberg orbit. The first step is to compare ordinary insertion with Garfinkle’s domino correspondence.

Extend $v \in W$ by $v(-i) = -v(i)$ and, when $X = B$, by $v(0) = 0$. Let $\hat{v}$ be its permutation of the ordered set

$$\begin{cases} 1, \dots, n, 0, -n, \dots, -1, & X = B, \\ 1, \dots, n, -n, \dots, -1, & X = C. \end{cases}$$

Write $P(\hat{v})$ for its ordinary Robinson–Schensted insertion tableau and $\text{sh } P(\hat{v})$ for the partition of its row lengths. We use the partition correspondence Π_X of Definition 4.3, with $\varepsilon = 1$ for $X = B$ and $\varepsilon = 0$ for $X = C$.

Lemma 8.1 (Insertion partitions). *For every $X \in \{B, C\}$ and every $v \in W$, the partition $\text{sh } P(\hat{v})$ belongs to $\mathcal{P}_\varepsilon(n)$. It is the common shape of the pair of domino tableaux assigned to v by Garfinkle’s correspondence [9, 10].*

Proof. Garfinkle’s correspondence associates to each signed permutation a pair of standard domino tableaux of the same shape [9, Theorem 1.2.13 and Remark 1.2.14]. These shapes lie in $\mathcal{P}_\varepsilon(n)$. In our coordinates, the reflection t of [7] changes the sign of coordinate n , and their s_i interchanges coordinates $n - i$ and $n - i + 1$. The order-preserving relabeling is $i \mapsto -(n + 1 - i)$, $-i \mapsto n + 1 - i$, and $0 \mapsto 0$ when $X = B$. For the embedding $v \mapsto \hat{v}$ above, the comparison with ordinary insertion in [7, Theorem 3.3] identifies this common Young diagram with the diagram of $P(\hat{v})$. This proves both assertions. $\square$

Write $\text{Irr}[\Lambda]$ for the set of isomorphism classes of irreducible constituents of a left-cell representation $[\Lambda]$.

Proposition 8.2 (Classical cell correspondence). *Let $X \in \{B, C\}$ and let Λ be an equal-parameter Kazhdan–Lusztig left cell in W , with the convention $a_s = s - 1$. Then the following statements hold.*

- (i) $[\Lambda]$ is multiplicity-free.
- (ii) The map

$$\Lambda \cap \Lambda^{-1} \longrightarrow \text{Irr}[\Lambda], \quad v \longmapsto \Pi_X(\text{sh } P(\hat{v}))$$

is a bijection.

Proof. The type- B and type- C cell theorem [17, Lemma 3.1 and Theorem 3.2] proves multiplicity-freeness and identifies the elements of a left–right cell intersection with the common irreducible constituents of the two cell representations. The identification sends an element to Π_X of its domino shape.

Take the right cell to be Λ^{-1} , whose representation has the same irreducible constituents as $[\Lambda]$. Lemma 8.1 identifies the domino shape of v with $\text{sh } P(\widehat{v})$, giving the stated bijection. $\square$

We now compare the classical and general linear nilradical intersections. Let V be the natural representation of $G^\vee$, with basis indexed by the ordered set above, and let $\mathfrak{n}_A$ be the strictly upper-triangular matrices in $\mathfrak{gl}(V)$. Choose the standard Borel so that $\mathfrak{n}^\vee = \mathfrak{g}^\vee \cap \mathfrak{n}_A$.

Lemma 8.3 (Geometric dominance bound). *For every $X \in \{B, C\}$ and every $v \in W$,*

$$(30) \quad \text{St}_{G^\vee}(v) \preceq \text{sh } P(\widehat{v}),$$

where a nilpotent orbit is identified with its partition.

Proof. Let p_v be the permutation matrix of $\widehat{v}$, and choose a monomial representative $\dot{v} \in G^\vee$ with the same underlying permutation. The two matrices differ by a diagonal matrix, which preserves every matrix-unit line. Consequently,

$$\text{Ad}(\dot{v})\mathfrak{n}_A = \text{Ad}(p_v)\mathfrak{n}_A, \quad \mathfrak{n}^\vee \cap \text{Ad}(\dot{v})\mathfrak{n}^\vee \subseteq \mathfrak{n}_A \cap \text{Ad}(p_v)\mathfrak{n}_A.$$

Put $\rho = \text{sh } P(\widehat{v})$. By the type- A Springer–Steinberg theorem [19], elements of Jordan type ρ form a dense subset of the general linear intersection. That intersection therefore lies in the closure of the general linear nilpotent orbit of type ρ . Every element of the classical intersection has Jordan partition dominated by ρ .

By Lemma 2.2, the classical intersection contains an element of $\text{St}_{G^\vee}(v)$. Applying the preceding bound to this element proves (30). $\square$

8.2. The canonical-cell representation. Let $\mathcal{O}^\vee = d_{BV}(\mathcal{O})$ have good parity. Choose an $\mathfrak{sl}_2$ -triple (ξ, h, η) with $\xi \in \mathcal{O}^\vee$ and h dominant. The orbit $\mathcal{O}^\vee$ is even: all eigenvalues of $\text{ad}(h)$ are even. Its Dynkin Levi subgroup is $L^\vee = Z_{G^\vee}(h)$. The Weyl group W_I is the standard parabolic subgroup generated by the simple reflections whose simple roots vanish on h . Let w_I be its longest element, and let Γ be the left cell containing w_I . Use the classes, representations, and partitions of Sections 5–7.

Proposition 8.4. *For either type $X \in \{B, C\}$ and the good-parity orbit $\mathcal{O}^\vee$ fixed above,*

$$[\Gamma] = \bigoplus_{J \subseteq \mathcal{R}} R_J.$$

This representation is multiplicity-free. For each $J \subseteq \mathcal{R}$ there is a unique $v_J \in \Gamma \cap \Gamma^{-1}$ such that $\Pi_X(\text{sh } P(\widehat{v_J})) = R_J$, and

$$(31) \quad \text{sh } P(\widehat{v_J}) = \widetilde{\rho}_J.$$

Proof. By [6, (5.27) and Proposition 5.28],

$$[\Gamma] \otimes \text{sgn} = \bigoplus_{x \in \overline{A}(\mathcal{O})} \sigma_{\mathcal{O}}(x, \mathbf{1}).$$

By Proposition 5.5 and (23), this gives the asserted multiplicity-free decomposition.

Proposition 8.2 identifies $\Gamma \cap \Gamma^{-1}$ bijectively with these constituents. For its element v_J , Lemma 5.3 gives

$$\Pi_X(\text{sh } P(\widehat{v_J})) = R_J = \Pi_X(\widetilde{\rho}_J).$$

The injectivity of Π_X [17, p. 80] proves (31). $\square$

Definition 8.5 (Canonical-cell labeling). For $x \in \overline{A}(\mathcal{O})$, write $x = x_J$ using (21), and set $v_x = v_J$ and $R_x = R_J$, where v_J is the element of Proposition 8.4. Equivalently, v_x is the unique element of $\Gamma \cap \Gamma^{-1}$ whose constituent under Proposition 8.2 is

$$R_x = \sigma_{\mathcal{O}}(x, \mathbf{1}) \otimes \text{sgn}, \quad \Pi_X(\text{sh } P(\widehat{v_x})) = R_x.$$

Thus $[\Gamma] = \bigoplus_{x \in \overline{A}(\mathcal{O})} R_x$.

8.3. Proof of the main theorem.

Proof of Theorem 1.1. Since $\mathcal{O}^\vee$ has good parity, the duality formula (14) determines $\mathcal{O} = d_{BV}(\mathcal{O}^\vee)$. Proposition 5.5 identifies all the classes x_J and proves occurrence of their labels $(x_J, \mathbf{1})$. Proposition 8.4 gives the cell decomposition and the unique elements $v_{x_J} = v_J$.

We apply Theorem 3.3 to the group $G^\vee$. Propositions 7.1 and 6.1 give

$$\text{Supp}_{G^\vee}(R_J) = \mathcal{O}_J^\vee = d_S(\mathcal{O}, x_J).$$

These orbits are pairwise distinct by Proposition 6.1. The geometric bound of Lemma 8.3 and (31) give

$$\text{St}_{G^\vee}(v_J) \preceq \tilde{\rho}_J.$$

The left-hand side is a type- X partition. Since $(\tilde{\rho}_J)_X = \mathcal{O}_J^\vee$, the defining maximality of X -collapse gives

$$\text{St}_{G^\vee}(v_J) \preceq \mathcal{O}_J^\vee, \quad \text{St}_{G^\vee}(v_J) \subseteq \overline{\mathcal{O}_J^\vee}.$$

All hypotheses of Theorem 3.3 have therefore been verified. It follows that

$$\text{St}_{G^\vee}(v_J) = \mathcal{O}_J^\vee = d_S(\mathcal{O}, x_J)$$

for every J , proving (2) for base dual orbits of good parity in both ordered pairs of groups.

Finally, $w_I^{-1} = w_I$, so $w_I = v_J$ for some $J \subseteq \mathcal{R}$. Since $\mathcal{O}^\vee$ is even, it is the Richardson orbit of the standard parabolic with Levi subgroup $L^\vee$ [6, Corollary 5.6]. The nilradical of this parabolic is $\mathfrak{n}^\vee \cap \text{Ad}(w_I)\mathfrak{n}^\vee$, so

$$\text{St}_{G^\vee}(w_I) = \mathcal{O}^\vee = \mathcal{O}_\emptyset^\vee.$$

The pairwise distinctness in Proposition 6.1 therefore gives $J = \emptyset$. Hence $v_0 = v_\emptyset = w_I$. $\square$

Corollary 8.6. *Let $w_0 = -1$ be the longest element of W . The left cell $\mathcal{C}_{\text{can}} = \Gamma w_0$ is pure for $\mathcal{O}$, and*

$$v_x w_0 \in \mathcal{C}_{\text{can}} \cap \mathcal{C}_{\text{can}}^{-1} \quad (x \in \overline{A}(\mathcal{O})).$$

Proof. Multiplication by w_0 takes a left cell to a left cell and tensors its representation with sign [17, Proposition 2.11]. The decomposition in Definition 8.5 therefore proves that $\mathcal{C}_{\text{can}}$ is pure for $\mathcal{O}$. Since w_0 is central and $w_0^2 = 1$,

$$(\Gamma w_0) \cap (\Gamma w_0)^{-1} = (\Gamma \cap \Gamma^{-1})w_0.$$

The asserted membership follows. $\square$

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