

DIRAC COHOMOLOGY OF STANDARD MODULES FOR REAL REDUCTIVE GROUPS

ABSTRACT. Let G be a real reductive group. We prove that normalized parabolic induction for G has zero Dirac cohomology whenever the inducing exponent is nonzero. In the case of standard modules, we showed that it has nonzero Dirac cohomology if and only if it is twisted-elliptic in the sense of Barbasch–Pandžić–Trapa. Both results can be seen as a real-group analogue of the work of Chan in the setup of graded affine Hecke algebras. Finally, we show that the same techniques can be applied to get a simplified proof of the generalized vanishing theorem for induced graded affine Hecke algebra modules.

1. INTRODUCTION

Let $G = \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a connected reductive linear algebraic group over $\mathbb{R}$. Fix a Cartan involution θ , put $K = G^\theta$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexified Cartan decomposition. If X is a Harish–Chandra $(\mathfrak{g}, K)$ -module and S is a spin module for the Clifford algebra of $\mathfrak{p}$, the Dirac operator D_X acts on $X \otimes S$. Its *Dirac cohomology* is

$$H_D(X) = \ker D_X / (\ker D_X \cap \operatorname{im} D_X).$$

The operator and the Clifford-algebra convention are defined in Section 3. The theorem of Huang–Pandžić [8, Theorem 2.3] states that, for an irreducible module with nonzero Dirac cohomology, the highest weight of a compact spin-cover constituent determines the infinitesimal character.

Let $P = MAN$ be a real parabolic subgroup with Langlands decomposition relative to θ , and put $\mathfrak{a} = \operatorname{Lie}(A) \otimes_{\mathbb{R}} \mathbb{C}$. For a smooth admissible representation σ of M and $\nu \in \mathfrak{a}^*$, let

$$I_P(\sigma, \nu) = \operatorname{Ind}_P^G(\sigma \otimes e^\nu)$$

denote normalized parabolic induction. Dirac cohomology is taken on the K -finite vectors. The normalization is specified in (3.4).

Theorem 1.1. *For every real parabolic subgroup $P = MAN$ of G , every smooth admissible representation σ of M , and every $\nu \in \mathfrak{a}^* \setminus \{0\}$,*

$$H_D(I_P(\sigma, \nu)) = 0.$$

The proof constructs, for each $H \in \operatorname{Lie}(A)$, an operator C_H on the compact realization such that

$$D_\nu C_H + C_H D_\nu = -2\nu(H) \operatorname{id}.$$

If $\nu(H) \neq 0$, this identity implies $\ker D_\nu \subseteq \operatorname{im} D_\nu$. The calculation applies to complex inducing exponents and does not require unitarity of σ .

Next, we apply Theorem 1.1 to standard modules. Such a module has a realization $Y = I_P(\sigma, \nu)$ in which σ is a discrete series or a limit of discrete series representation and $\operatorname{Re} \nu$ lies in the closed positive chamber. Theorem 1.1 implies that $H_D(Y) \neq 0$ forces $\nu = 0$. In other words, Y is irreducible and tempered.

A $(\mathfrak{g}, K)$ -module is called *twisted-elliptic* if it admits an extension to $K^+ = K \times \langle \vartheta \rangle$, where ϑ implements θ , with nonzero twisted Dirac index (see Section 3.2). The following theorem can be seen as an analog of [4, Theorem 1.1] for real groups:

Theorem 1.2. *Let Y be a Langlands standard $(\mathfrak{g}, K)$ -module. Then*

$$(1.1) \quad H_D(Y) \neq 0 \iff Y \text{ is tempered, twisted-elliptic.}$$

Key words and phrases. Dirac cohomology, twisted Dirac index, standard modules, graded affine Hecke algebras.

The proof combines Theorem 1.1 with the classification of tempered representations with nonzero Dirac cohomology in [6, Theorems 6.1 and 7.5] and the twisted-index formula of [3, Theorem 5.7].

In Section 4, we prove that the same anticommutator identity holds for parabolically induced Hecke modules. Consequently, it recovers and simplifies the non-tempered vanishing assertion of [4].

2. AN ALGEBRAIC VANISHING CRITERION

Let $\mathcal{V}$ be a complex vector space and let $D \in \text{End}_{\mathbb{C}}(\mathcal{V})$. Define

$$\mathcal{H}_D(\mathcal{V}) = \ker D / (\ker D \cap \text{im } D).$$

Lemma 2.1. *Suppose that $C, T \in \text{End}_{\mathbb{C}}(\mathcal{V})$ satisfy*

$$DC + CD = -2T, \quad DT = TD,$$

and that T is invertible. Then $\mathcal{H}_D(\mathcal{V}) = 0$. In particular, this holds if $DC + CD = -2a \text{id}_{\mathcal{V}}$ for some $a \in \mathbb{C} \setminus \{0\}$.

Proof. Since T^{-1} commutes with D , the operator $Q = -\frac{1}{2}CT^{-1}$ satisfies

$$DQ + QD = -\frac{1}{2}(DC + CD)T^{-1} = \text{id}_{\mathcal{V}}.$$

For every $x \in \ker D$ we therefore have $x = DQx$. Thus $\ker D \subseteq \text{im } D$. The scalar assertion follows by taking $T = a \text{id}_{\mathcal{V}}$. $\square$

3. REAL REDUCTIVE GROUPS

3.1. Dirac cohomology and the infinitesimal character. Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\mathbb{R}$, and put $G = \mathbf{G}(\mathbb{R})$. Fix a Cartan involution θ and write

$$K = G^{\theta}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

The subscript 0 denotes a real Lie algebra, and its omission denotes complexification. The group G need not be connected. Choose an $\text{Ad}(G)$ -invariant and θ -invariant symmetric bilinear form B such that

$$B_{\theta}(U, V) = -B(U, \theta V)$$

is positive definite on $\mathfrak{g}_0$. Extend B complex linearly.

Let $C(\mathfrak{p})$ be the complex Clifford algebra with defining relation

$$UV + VU = -2B(U, V)1 \quad (U, V \in \mathfrak{p}).$$

Fix an irreducible $C(\mathfrak{p})$ -module S . Let $\widetilde{K}$ be the pullback of $\text{Pin}(\mathfrak{p}_0) \rightarrow O(\mathfrak{p}_0)$ under $\text{Ad}|_{\mathfrak{p}_0} : K \rightarrow O(\mathfrak{p}_0)$. Write $c : C(\mathfrak{p}) \rightarrow \text{End}_{\mathbb{C}}(S)$ for the Clifford action. For a Harish–Chandra $(\mathfrak{g}, K)$ -module X , with Lie-algebra action π_X , define the *Dirac operator* by

$$D_X = \sum_{j=1}^{\dim \mathfrak{p}_0} \pi_X(Z_j) \otimes c(Z_j),$$

where (Z_j) is a B -orthonormal basis of $\mathfrak{p}_0$. This operator is independent of the chosen orthonormal basis. Let $s_S(\tilde{k})$ denote the spin action of $\tilde{k} \in \widetilde{K}$ lying over $k \in K$. The diagonal action is

$$\tilde{\pi}_X(\tilde{k}) = \pi_X(k) \otimes s_S(\tilde{k}).$$

Define the *Dirac cohomology* of X by:

$$H_D(X) = \ker D_X / (\ker D_X \cap \text{im } D_X).$$

Since the diagonal action of $\widetilde{K}$ preserves both the numerator and the denominator, $H_D(X)$ is a $\widetilde{K}$ -module.

Fix a maximal torus T_0 of K° , with Lie algebra $\mathfrak{t}_0$. Its centralizer $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ in $\mathfrak{g}_0$ is a fundamental θ -stable Cartan subalgebra; here $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Fix positive roots for $(\mathfrak{k}, \mathfrak{t})$ and let $\rho_c \in \mathfrak{t}^*$ be their half-sum. Regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extension by zero on $\mathfrak{a}$. An

infinitesimal character is called *real* if its representative in $\mathfrak{h}^*$ lies in $i\mathfrak{t}_0^* \oplus \mathfrak{a}_0^*$. This condition is Weyl invariant and independent of the chosen θ -stable Cartan; see [6, pp. 43–44].

Theorem 3.1 (Huang–Pandžić). *Let X be an irreducible Harish–Chandra module with infinitesimal character represented by $\Lambda \in \mathfrak{h}^*$. If an irreducible $\widetilde{K}^\circ$ -constituent of $H_D(X)$ has highest weight γ , then*

$$\Lambda \in W(\mathfrak{g}, \mathfrak{h})(\gamma + \rho_c),$$

where $W(\mathfrak{g}, \mathfrak{h})$ is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. In particular, X has real infinitesimal character.

The algebraic identity underlying this theorem is [8, Corollary 3.5], with the disconnected formulation in [6, Theorem 2.1]. Let $\Delta : U(\mathfrak{k}) \rightarrow U(\mathfrak{g}) \otimes C(\mathfrak{p})$ be the diagonal embedding defined by the inclusion $\mathfrak{k} \subseteq \mathfrak{g}$ and the infinitesimal spin action on $C(\mathfrak{p})$. On $X \otimes S$, its action is

$$\Delta(Z)|_{X \otimes S} = \pi_X(Z) \otimes \text{id}_S + \text{id}_X \otimes s'_S(Z) \quad (Z \in \mathfrak{k}),$$

where s'_S is the complexified differential of s_S . For every $z \in Z(U(\mathfrak{g}))$, there is an element a_z of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ such that

$$z \otimes 1 - \Delta(\zeta(z)) = Da_z + a_z D,$$

where $\zeta : Z(U(\mathfrak{g})) \rightarrow Z(U(\mathfrak{k}))^K$ is the homomorphism which, under the Harish–Chandra isomorphisms, is restriction from $\mathfrak{h}^*$ to $\mathfrak{k}^*$ [8, Theorem 5.5].

3.2. Standard modules and the twisted Dirac index. Put

$$G^+ = G \rtimes \langle \vartheta \rangle, \quad \vartheta^2 = 1, \quad \vartheta g \vartheta^{-1} = \theta(g), \quad K^+ = K \times \langle \vartheta \rangle.$$

A $(\mathfrak{g}, K^+)$ -module is a $(\mathfrak{g}, K)$ -module (π, X) equipped with an involution $\pi(\vartheta)$ satisfying

$$\pi(\vartheta)\pi(k) = \pi(k)\pi(\vartheta), \quad \pi(\vartheta)\pi(Z) = \pi(\theta Z)\pi(\vartheta) \quad (k \in K, Z \in \mathfrak{g}).$$

Write $X^\pm$ for the eigenspaces of $\pi(\vartheta)$. Since $\theta|_{\mathfrak{p}} = -1$, the operator $\pi(\vartheta) \otimes \text{id}_S$ anticommutes with D_X . It therefore induces an involution on $H_D(X)$; denote its eigenspaces by $H_D(X)^\pm$.

Definition 3.2. For a finite-length $(\mathfrak{g}, K^+)$ -module X with infinitesimal character, its *twisted Dirac index* is

$$I_\theta(X) = [H_D(X)^+] - [H_D(X)^-] \in R(\widetilde{K}),$$

where $R(\widetilde{K})$ denotes the Grothendieck group of finite-dimensional representations of $\widetilde{K}$. A $(\mathfrak{g}, K)$ -module Y is *twisted-elliptic* if it admits an extension X to $(\mathfrak{g}, K^+)$ with $I_\theta(X) \neq 0$.

By [3, Proposition 1.8 and equation (1.13)],

$$I_\theta(X) = [X^+ \otimes S] - [X^- \otimes S],$$

where the right-hand side is initially interpreted in the Grothendieck group of admissible $\widetilde{K}$ -modules.

We now related the twisted Dirac index with the ordinary Dirac index when $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}$. Then $\dim \mathfrak{p}$ is even, and $S = S^+ \oplus S^-$ has its half-spin grading. Let Γ act by ± 1 on $S^\pm$, and put

$$\widetilde{K}_{\text{ev}} = \{(k, u) \in \widetilde{K} : u \in \text{Spin}(\mathfrak{p}_0)\}.$$

For a finite-length $(\mathfrak{g}, K)$ -module Y with infinitesimal character, let $H_D(Y)_{\text{spin}}^\pm$ be the eigenspaces of $\text{id}_Y \otimes \Gamma$. Its *ordinary Dirac index* is

$$I_D(Y) = [H_D(Y)_{\text{spin}}^+] - [H_D(Y)_{\text{spin}}^-] = [Y \otimes S^+] - [Y \otimes S^-] \in R(\widetilde{K}_{\text{ev}}).$$

If K is connected, then $\widetilde{K}_{\text{ev}} = \widetilde{K}$.

To relate it with twisted Dirac index, one can extend X into a $(\mathfrak{g}, K^+)$ -module Y such that $X \cong Y$ as $(\mathfrak{g}, K^+)$ -modules. More precisely, suppose $\theta = \text{Ad}(k_0)$ for some $k_0 \in K$, then for a choice of the lift $\tilde{k}_0 \in \widetilde{K}_{\text{ev}}$ of k_0 normalized as in [3, §1.4], the index characters satisfy

$$(3.1) \quad \chi_{I_\theta(X)}(\tilde{k}) = \chi_{I_D(Y)}(\tilde{k}_0 \tilde{k}) \quad (\tilde{k} \in \widetilde{K}_{\text{ev}}),$$

where χ denotes the character of a virtual representation.

Definition 3.3. A *Langlands standard* $(\mathfrak{g}, K)$ -module is the Harish–Chandra module of a normalized induced representation

$$Y = I_P(\sigma, \nu), \quad P = MAN,$$

where M admits a compact Cartan subgroup, σ is an irreducible discrete series or limit of discrete series representation of M , and

$$\langle \operatorname{Re} \nu, \alpha^\vee \rangle \geq 0 \quad (\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})).$$

Here $\Sigma(\mathfrak{n}, \mathfrak{a})$ is the set of restricted roots occurring in $\mathfrak{n}$. We require the associated Cartan datum to be a final limit character in the sense of [1, Definition 11.13].

By [1, Theorem 11.14], a standard module has a unique irreducible quotient, its *Langlands quotient*. If $\nu = 0$, the inducing representation is unitary and tempered, and Y is a finite direct sum of irreducible tempered modules [9, Theorem 11.198(a)]. The uniqueness of its irreducible quotient therefore implies that Y is irreducible.

Now we express a standard module $Y = I_P(\sigma, \nu)$ in the notations of [3, Section 5.6]. Choose a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$, put $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, and let $H = TA$ be the corresponding Cartan subgroup, with $T = H \cap K$. Choose a θ -stable Borel subalgebra $\mathfrak{b}_M = \mathfrak{t} \oplus \mathfrak{u}_M$ and a character $\xi : T \rightarrow \mathbb{C}^\times$ such that

$$\sigma \simeq \mathcal{L}_{\mathfrak{b}_M, s_M}^M(\mathbb{C}_\xi), \quad s_M = \dim(\mathfrak{u}_M \cap \mathfrak{k}).$$

The differential $d\xi$ is an element of $\mathfrak{t}^*$. With

$$\rho_M = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}_M, \mathfrak{t})} \alpha, \quad \Lambda_\sigma = d\xi + \rho_M,$$

the parameter Λ_σ represents the infinitesimal character of σ .

Let $\mathfrak{n}^- = \theta\mathfrak{n}$, and put

$$\mathfrak{b} = (\mathfrak{b}_M \oplus \mathfrak{a}) \oplus \mathfrak{n}^- = \mathfrak{h} \oplus \mathfrak{u}, \quad \mathfrak{u} = \mathfrak{u}_M \oplus \mathfrak{n}^-.$$

We use the left-derived cohomological-induction convention of [9, (5.2)–(5.3) and §XI.2]: $\mathcal{L}_{\mathfrak{b}, j}$ is obtained by algebraic induction from the opposite Borel $\mathfrak{b}^{\text{op}}$, with the twist $\bigwedge^{\text{top}} \mathfrak{u}$, followed by the j -th left-derived Bernstein functor.

Define

$$\rho_P(Z) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}(Z)|_{\mathfrak{n}}) \quad (Z \in \mathfrak{a}), \quad \epsilon_P(t) = \det(\operatorname{Ad}(t)|_{\mathfrak{n}}) \quad (t \in T).$$

The character ϵ_P takes values in $\{\pm 1\}$: T acts on the real vector space $\mathfrak{n}_0$, and compactness forces the absolute value of the determinant to be one. In particular, $d\epsilon_P = 0$. The required character of H is

$$(3.2) \quad \lambda(t \exp Z) = \xi(t) \epsilon_P(t)^{-1} e^{(\nu + \rho_P)(Z)} \quad (t \in T, Z \in \mathfrak{a}_0).$$

Thus

$$d\lambda|_{\mathfrak{t}} = \Lambda_\sigma - \rho_M, \quad d\lambda|_{\mathfrak{a}} = \nu + \rho_P.$$

Extend $\mathbb{C}_\xi$ and $\bigwedge^{\text{top}} \mathfrak{u}_M$ trivially across $\mathfrak{a}$. If $\mathbb{C}_{\nu - \rho_P}$ has trivial $(\mathfrak{t}, T)$ -action and $\mathfrak{a}$ -action $\nu - \rho_P$, then

$$\mathbb{C}_\lambda \otimes \bigwedge^{\text{top}} \mathfrak{u} \simeq (\mathbb{C}_\xi \otimes \bigwedge^{\text{top}} \mathfrak{u}_M) \otimes \mathbb{C}_{\nu - \rho_P}.$$

Induction in stages and [9, Proposition 11.65 and Theorem 11.77] therefore give

$$(3.3) \quad Y \simeq \mathcal{L}_{\mathfrak{b}, s}(\mathbb{C}_\lambda) =: X_{\mathfrak{b}}(\lambda), \quad s = \dim(\mathfrak{u} \cap \mathfrak{k}) = s_M$$

as in the notation of [3, Definition 5.5].

We now explain how to extend a standard $(\mathfrak{g}, K)$ -module Y into a $(\mathfrak{g}, K^+)$ -module. Firstly, note that the Borel subalgebra in (3.3) need not be θ -stable. Define the twist Y^θ to have the same K -action as Y and Lie-algebra action $\pi_{Y^\theta}(Z) = \pi_Y(\theta Z)$. Therefore

$$Y^\theta \simeq X_{\theta\mathfrak{b}}(\theta\lambda), \quad (\theta\lambda)(h) = \lambda(\theta h).$$

If $Y^\theta \simeq Y$, write Y^ϵ , $\epsilon = \pm 1$, for the two standard extensions specified in [3, §5.2]. If $Y^\theta \not\simeq Y$, the module $\tilde{Y} = Y \oplus Y^\theta$ extends to $(\mathfrak{g}, K^+)$ by $\vartheta(x, y) = (y, x)$.

For a θ -stable Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$, we also write $A_{\mathfrak{b}}(\lambda) = X_{\mathfrak{b}}(\lambda)$. The character λ is *weakly good* if

$$\operatorname{Re}\langle d\lambda + \rho(\mathfrak{u}), \alpha^\vee \rangle \geq 0 \quad (\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})).$$

If $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is fundamental, then $\mathfrak{u} \cap \mathfrak{k}$ determines positive roots for $(\mathfrak{k}, \mathfrak{t})$. Write $\rho(\mathfrak{u} \cap \mathfrak{p}) \in \mathfrak{t}^*$ for half the sum, with multiplicities, of the $\mathfrak{t}$ -weights in $\mathfrak{u} \cap \mathfrak{p}$.

The following result follows directly from Theorem 5.7 of [3]:

Theorem 3.4 (Barbasch–Pandžić–Trapa). *Let Y be a Langlands standard $(\mathfrak{g}, K)$ -module.*

- (i) *Suppose that $Y^\theta \simeq Y$. Then $I_\theta(Y^\varepsilon) \neq 0$ if and only if Y admits a realization $A_{\mathfrak{b}}(\lambda)$ with $\mathfrak{b}$ a θ -stable Borel containing a fundamental Cartan, λ weakly good, $\theta\lambda = \lambda$, and $d\lambda|_{\mathfrak{t}} + \rho(\mathfrak{u} \cap \mathfrak{p})$ is dominant for the positive compact roots.*
- (ii) *If $Y^\theta \not\simeq Y$, then $I_\theta(\tilde{Y}) = 0$.*

In the equal-rank case, one always has $Y \cong Y^\theta$. The ordinary-index pairing also admits the elliptic-pairing interpretation of [3, Theorem 6.5] and [7].

3.3. The Dirac family. Let $P = MAN$ be a real parabolic subgroup, with Langlands decomposition relative to θ . In this subsection, put

$$\mathfrak{m}_0 = \operatorname{Lie}(M), \quad \mathfrak{a}_0 = \operatorname{Lie}(A), \quad \mathfrak{n}_0 = \operatorname{Lie}(N).$$

Here $\mathfrak{a}_0$ denotes the split component of the inducing parabolic. Define

$$\rho_P(H) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}_0}) \quad (H \in \mathfrak{a}_0).$$

For a smooth admissible representation σ on V_σ and $\nu \in \mathfrak{a}^*$, normalized induction consists of smooth functions $f : G \rightarrow V_\sigma$ satisfying

$$(3.4) \quad \begin{aligned} f(gman) &= e^{-(\nu + \rho_P)(\log a)} \sigma(m)^{-1} f(g), \\ (\pi_\nu(g_0)f)(g) &= f(g_0^{-1}g). \end{aligned}$$

The variables satisfy $g, g_0 \in G$, $m \in M$, $a \in A$, and $n \in N$. If σ is specified as a finite-length Harish–Chandra module, choose a smooth admissible globalization and take the K -finite vectors after induction.

Restriction to K identifies the K -finite vectors of $I_P(\sigma, \nu) \otimes S$ with the space $\mathcal{F}_\sigma$ of smooth, left K -finite functions $F : K \rightarrow V_\sigma \otimes S$ satisfying

$$(3.5) \quad F(km) = (\sigma(m)^{-1} \otimes \operatorname{id}_S) F(k) \quad (k \in K, m \in K \cap M).$$

This space is independent of ν . The spin factor is external: in the extension of F to G , the right transformation law acts on V_σ and trivially on S .

For $\tilde{k}_0 \in \tilde{K}$ lying over $k_0 \in K$, the diagonal action $\tilde{\pi}_\nu(\tilde{k}_0) = \pi_\nu(k_0) \otimes s_S(\tilde{k}_0)$ is given by

$$(\tilde{\pi}_\nu(\tilde{k}_0)F)(k) = (\operatorname{id}_{V_\sigma} \otimes s_S(\tilde{k}_0))F(k_0^{-1}k).$$

Its finite vectors coincide with the finite vectors for external left translation, since S is finite dimensional. Write D_ν for the Dirac operator on $\mathcal{F}_\sigma$. The Lie-algebra action on $I_P(\sigma, \nu) \otimes S$ is $\pi_\nu(Z) \otimes \operatorname{id}_S$ for $Z \in \mathfrak{g}$. For $U \in \mathfrak{p}$, the Clifford action $\operatorname{id}_{I_P(\sigma, \nu)} \otimes c(U)$ is given in the compact realization by

$$((\operatorname{id}_{I_P(\sigma, \nu)} \otimes c(U))F)(k) = (\operatorname{id}_{V_\sigma} \otimes c(U))F(k).$$

Thus, for any B -orthonormal basis (Z_j) of $\mathfrak{p}_0$,

$$(D_\nu F)(k) = \sum_j (\operatorname{id}_{V_\sigma} \otimes c(Z_j))((\pi_\nu(Z_j) \otimes \operatorname{id}_S)F)(k).$$

Lemma 3.5 (Anticommutator identity). *For every $H \in \mathfrak{a}_0$, the formula*

$$(C_H F)(k) = (\operatorname{id}_{V_\sigma} \otimes c(\operatorname{Ad}(k)H))F(k)$$

defines an endomorphism of $\mathcal{F}_\sigma$, and

$$(3.6) \quad D_\nu C_H + C_H D_\nu = -2\nu(H) \operatorname{id}_{\mathcal{F}_\sigma}.$$

Proof. For $m \in K \cap M$, one has $\text{Ad}(m)H = H$ because M centralizes $\mathfrak{a}$. Hence

$$\begin{aligned} (C_H F)(km) &= (\text{id}_{V_\sigma} \otimes c(\text{Ad}(km)H))(\sigma(m)^{-1} \otimes \text{id}_S)F(k) \\ &= (\sigma(m)^{-1} \otimes \text{id}_S)(\text{id}_{V_\sigma} \otimes c(\text{Ad}(k)H))F(k) \\ &= (\sigma(m)^{-1} \otimes \text{id}_S)(C_H F)(k). \end{aligned}$$

The second equality uses both $\text{Ad}(m)H = H$ and the commutation of operators on distinct tensor factors. Thus $C_H F$ satisfies (3.5). The matrix entries of $k \mapsto c(\text{Ad}(k)H)$ belong to a finite-dimensional space of matrix coefficients of the adjoint representation of K . Their products with K -finite functions are K -finite. Therefore C_H preserves $\mathcal{F}_\sigma$.

Let $\widehat{F}$ be the extension to G defined by (3.4), with trivial action on the spin factor. For $g = kman$,

$$(3.7) \quad \widehat{C_H F}(g) = (\text{id}_{V_\sigma} \otimes c(\text{Ad}(k)H))\widehat{F}(g).$$

The preceding equivariance calculation makes this formula independent of the choice of the compact factor k .

Let κ be projection onto $\mathfrak{k}_0$ in the direct-sum decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus (\mathfrak{m}_0 \cap \mathfrak{p}_0) \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0.$$

Fix $k \in K$, and choose a B -orthonormal basis $Z_1, \dots, Z_d$ of $\mathfrak{p}_0$. Then $\text{Ad}(k)Z_1, \dots, \text{Ad}(k)Z_d$ is also orthonormal. Near $t = 0$, choose a factorization

$$\exp(-tZ_j) = k_j(t)m_j(t)a_j(t)n_j(t)$$

whose factors equal the identity at $t = 0$ and for which $m'_j(0) \in \mathfrak{m}_0 \cap \mathfrak{p}_0$. It follows that $k'_j(0) = \kappa(-Z_j)$ and

$$\exp(-t \text{Ad}(k)Z_j)k = k k_j(t)m_j(t)a_j(t)n_j(t).$$

The product rule, applied to (3.7), gives

$$(3.8) \quad \begin{aligned} ((\pi_\nu(\text{Ad}(k)Z_j) \otimes \text{id}_S)(C_H F))(k) &= (\text{id}_{V_\sigma} \otimes c(\text{Ad}(k)[\kappa(-Z_j), H]))F(k) \\ &\quad + (\text{id}_{V_\sigma} \otimes c(\text{Ad}(k)H))((\pi_\nu(\text{Ad}(k)Z_j) \otimes \text{id}_S)F)(k). \end{aligned}$$

Here $\sigma(m_j(t)) \otimes \text{id}_S$ commutes with $\text{id}_{V_\sigma} \otimes c(U)$ for every $U \in \mathfrak{p}$.

At the fixed point k , use the orthonormal basis $(\text{Ad}(k)Z_j)$ to write D_ν . The Clifford relations in (3.8) then give

$$(3.9) \quad \begin{aligned} ((D_\nu C_H + C_H D_\nu)F)(k) &= \sum_j \left((\text{id}_{V_\sigma} \otimes c(\text{Ad}(k)Z_j)c(\text{Ad}(k)[\kappa(-Z_j), H]))F(k) \right. \\ &\quad \left. - 2((\pi_\nu(\text{Ad}(k)H) \otimes \text{id}_S)F)(k) \right). \end{aligned}$$

Since $\exp(-t \text{Ad}(k)H)k = k \exp(-tH)$, the transformation law (3.4) implies

$$(3.10) \quad -2((\pi_\nu(\text{Ad}(k)H) \otimes \text{id}_S)F)(k) = -2(\nu + \rho_P)(H)F(k).$$

The first sum in (3.9) is independent of the orthonormal basis, since it is the image of $\sum_j Z_j \otimes Z_j$ under the bilinear map

$$U \otimes V \mapsto \text{id}_{V_\sigma} \otimes \left(c(\text{Ad}(k)U)c(\text{Ad}(k)[\kappa(-V), H]) \right).$$

We evaluate it using the restricted-root decomposition of $\mathfrak{n}_0$. For each $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$, choose a B_θ -orthonormal basis $X_{\alpha,1}, \dots, X_{\alpha,m_\alpha}$ of the corresponding real root space, and put

$$Z_{\alpha,r} = \frac{X_{\alpha,r} - \theta X_{\alpha,r}}{\sqrt{2}}, \quad T_{\alpha,r} = \frac{X_{\alpha,r} + \theta X_{\alpha,r}}{\sqrt{2}}.$$

Together with orthonormal bases of $\mathfrak{m}_0 \cap \mathfrak{p}_0$ and $\mathfrak{a}_0$, the vectors $Z_{\alpha,r}$ form an orthonormal basis of $\mathfrak{p}_0$. Moreover,

$$-Z_{\alpha,r} = T_{\alpha,r} - \sqrt{2}X_{\alpha,r}, \quad \kappa(-Z_{\alpha,r}) = T_{\alpha,r}, \quad [T_{\alpha,r}, H] = -\alpha(H)Z_{\alpha,r}.$$

Consequently,

$$\mathrm{id}_{V_\sigma} \otimes \left(c(\mathrm{Ad}(k)Z_{\alpha,r})c(\mathrm{Ad}(k)[T_{\alpha,r}, H]) \right) = \alpha(H) \mathrm{id}_{V_\sigma \otimes S}.$$

The compact projection of a vector in $(\mathfrak{m}_0 \cap \mathfrak{p}_0) \oplus \mathfrak{a}_0$ is zero. Thus the first sum in (3.9) is

$$\left(\sum_{\alpha} m_{\alpha} \alpha(H) \right) F(k) = 2\rho_P(H)F(k).$$

Combining this with (3.10) proves (3.6). $\square$

Corollary 3.6. *If $\nu \neq 0$, then*

$$H_D(I_P(\sigma, \nu)) = 0.$$

Proof of Theorem 1.1 and Corollary 3.6. Since $\mathfrak{a}_0$ spans $\mathfrak{a}$ over $\mathbb{C}$, there exists $H \in \mathfrak{a}_0$ with $\nu(H) \neq 0$. Apply Lemma 2.1 to the identity in Lemma 3.5, with $a = \nu(H)$. $\square$

Remark 3.7 (Dependence on the inducing exponent). Extend $H \mapsto C_H$ complex linearly from $\mathfrak{a}_0$ to $\mathfrak{a}$. Let $\nu^\sharp \in \mathfrak{a}$ be defined by $B(\nu^\sharp, H) = \nu(H)$ for $H \in \mathfrak{a}$. Differentiating the inducing character gives

$$D_\nu = D_0 + C_{\nu^\sharp}.$$

Indeed, for $Z \in \mathfrak{g}$ and $k \in K$,

$$(((\pi_\nu(Z) - \pi_0(Z)) \otimes \mathrm{id}_S)F)(k) = \nu(\mathrm{pr}_{\mathfrak{a}}(\mathrm{Ad}(k^{-1})Z))F(k),$$

where $\mathrm{pr}_{\mathfrak{a}}$ denotes the projection in the complexification of the decomposition defining κ . The projection onto $\mathfrak{a}$ restricts on $\mathfrak{p}$ to the B -orthogonal projection, so contraction with the Clifford factor is $\mathrm{id}_{V_\sigma} \otimes c(\mathrm{Ad}(k)\nu^\sharp)$. The pointwise Clifford relations and Lemma 3.5 at $\nu = 0$ give

$$D_\nu^2 = D_0^2 - B(\nu^\sharp, \nu^\sharp) \mathrm{id}_{\mathcal{F}_\sigma}.$$

If σ is unitary, then D_0 is symmetric on the K -finite vectors. Hence $\ker D_0 \cap \mathrm{im} D_0 = 0$, and $H_D(I_P(\sigma, 0)) \simeq \ker D_0$. For $\nu \neq 0$, Corollary 3.6 already gives vanishing, even when $B(\nu^\sharp, \nu^\sharp) = 0$.

Remark 3.8. For $\tilde{k} \in \tilde{K}$ lying over $k \in K$, put $\epsilon(k) = \det(\mathrm{Ad}(k)|_{\mathfrak{p}_0})$. The Pin convention gives

$$\tilde{\pi}_\nu(\tilde{k})D_\nu\tilde{\pi}_\nu(\tilde{k})^{-1} = \epsilon(k)D_\nu, \quad \tilde{\pi}_\nu(\tilde{k})C_H\tilde{\pi}_\nu(\tilde{k})^{-1} = \epsilon(k)C_H.$$

The proof of Lemma 3.5 is a pointwise calculation and therefore applies also when K has orientation-reversing elements.

3.4. The twisted index and Dirac cohomology of standard modules. For a standard module $Y = I_P(\sigma, \nu)$, nonzero Dirac cohomology forces $\nu = 0$ by Corollary 3.6. Definition 3.3 then implies that Y is irreducible and tempered. We use the following consequence of [6, Theorems 6.1 and 7.5].

Theorem 3.9 (Dong–Huang). *Let Y be an irreducible tempered $(\mathfrak{g}, K)$ -module. Then $H_D(Y) \neq 0$ if and only if there exist a θ -stable Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$, with fundamental Cartan subgroup $H = TA$, and a weakly good unitary character λ of H such that*

$$Y \simeq A_{\mathfrak{b}}(\lambda), \quad \theta\lambda = \lambda,$$

and

$$(3.11) \quad d\lambda|_{\mathfrak{t}} + \rho(\mathfrak{u} \cap \mathfrak{p}) \quad \text{is dominant for the positive compact roots.}$$

The positive compact roots are those determined by $\mathfrak{u} \cap \mathfrak{k}$.

Proof. If $H_D(Y) \neq 0$, [6, Theorem 7.5] gives the indicated cohomological-induction realization. For a Borel subalgebra, the Levi factor is the fundamental Cartan. The nonvanishing criterion in [6, Theorem 6.1] therefore reduces to (3.11), with $\theta\lambda = \lambda$. Conversely, the same theorem gives nonzero Dirac cohomology for a realization satisfying these conditions. $\square$

Proof of Theorem 1.2. Suppose that $H_D(Y) \neq 0$. The preceding reduction makes Y irreducible and tempered. Theorem 3.9 provides a θ -stable realization satisfying the compact dominance condition. Theorem 3.4 then gives $I_\theta(Y^\varepsilon) \neq 0$. Conversely, nonzero twisted Dirac index implies nonzero Dirac cohomology by its definition. In either case the infinitesimal character is real by Theorem 3.1. $\square$

4. APPLICATION TO GRADED AFFINE HECKE ALGEBRAS

We apply Lemma 2.1 to parabolically induced modules for graded affine Hecke algebras. The resulting vanishing theorem is the counterpart of Theorem 1.1. The final subsection applies it to geometric standard modules.

4.1. Definitions and normalization. Let $V_{\mathbb{R}}$ be a nonzero finite-dimensional real Euclidean space with inner product B , and let $R \subset V_{\mathbb{R}}$ be a reduced root system spanning $V_{\mathbb{R}}$. Let W be its Weyl group. Choose simple roots Π and positive roots R^+ . Put

$$V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \quad \alpha^\vee = \frac{2\alpha}{B(\alpha, \alpha)}.$$

The form B is extended complex bilinearly. These spaces and this form are independent of the Lie-algebra notation in Section 3.

Fix a W -invariant function $k : R \rightarrow \mathbb{C}$. The graded affine Hecke algebra $\mathbb{H}$ is generated by $S(V)$ and $\mathbb{C}[W]$, with standard group elements t_w , subject to

$$(4.1) \quad t_{s_\alpha} v - s_\alpha(v) t_{s_\alpha} = k_\alpha B(v, \alpha^\vee) \quad (\alpha \in \Pi, v \in V).$$

Its Poincaré–Birkhoff–Witt decomposition is $\mathbb{H} \simeq \mathbb{C}[W] \otimes S(V)$ as a vector space. For a nonreduced presentation, choose one reduced root system with the same reflection hyperplanes and use the coefficients determined by (4.1). More precisely, replacing a simple root α by $b_\alpha \alpha$, with $b_\alpha > 0$, replaces k_α by $b_\alpha k_\alpha$. The product $k_\alpha \alpha^\vee$ is unchanged. All sums below are over the positive roots of the chosen reduced system, with one term for each reflection.

For $v \in V$, define

$$(4.2) \quad \tilde{v} = v - A(v), \quad A(v) = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha B(v, \alpha^\vee) t_{s_\alpha}.$$

Let $C(V)$ be the Clifford algebra with relations

$$vu + uv = -2B(v, u)1 \quad (v, u \in V),$$

and choose an irreducible $C(V)$ -module $S_{\mathbb{H}}$, with action $c : C(V) \rightarrow \text{End}_{\mathbb{C}}(S_{\mathbb{H}})$. For a B -orthonormal basis $e_1, \dots, e_n$ of V , put

$$(4.3) \quad D_{\mathbb{H}} = \sum_{i=1}^n \tilde{e}_i \otimes c(e_i).$$

This element of $\mathbb{H} \otimes \text{End}_{\mathbb{C}}(S_{\mathbb{H}})$ is independent of the orthonormal basis. If E is an $\mathbb{H}$ -module with action π_E , then $D_{\mathbb{H}}$ acts on $E \otimes S_{\mathbb{H}}$ as

$$D_{\mathbb{H}}|_{E \otimes S_{\mathbb{H}}} = \sum_i \pi_E(\tilde{e}_i) \otimes c(e_i).$$

Thus $h \in \mathbb{H}$ acts as $\pi_E(h) \otimes \text{id}_{S_{\mathbb{H}}}$, and $v \in V \subseteq C(V)$ acts as $\text{id}_E \otimes c(v)$. Define

$$H_D(E) = \ker D_{\mathbb{H}} / (\ker D_{\mathbb{H}} \cap \text{im } D_{\mathbb{H}}),$$

with kernel and image taken in $E \otimes S_{\mathbb{H}}$. The modified generators are those of [2, (2.13)], and (4.3) is the Dirac element of [2, Definition 3.1], with the root-space convention of [4, §2.3]. The polynomial space denoted by $V^\vee$ in [2] corresponds to V here after identifying the dual Euclidean spaces.

4.2. Parabolic induction and the anticommutator. For $J \subseteq \Pi$, let W_J be generated by $\{s_\alpha : \alpha \in J\}$, and let $\mathbb{H}_J$ be the subalgebra generated by all of $S(V)$ and $\mathbb{C}[W_J]$. Define

$$\mathfrak{a}_J = V^{W_J}.$$

Every $H \in \mathfrak{a}_J$ is central in $\mathbb{H}_J$: it commutes with $S(V)$, and (4.1) gives $t_{s_\alpha}H = Ht_{s_\alpha}$ for $\alpha \in J$.

Lemma 4.1 ([2, Proposition 2.10(2)]). *For every $v \in V$ and $w \in W$,*

$$\tilde{v}t_w = t_w \widetilde{w^{-1}v}.$$

Proof. For $\alpha \in \Pi$, the reflection s_α permutes $R^+ \setminus \{\alpha\}$. The definition of $A(v)$ gives

$$t_{s_\alpha}A(v)t_{s_\alpha} = A(s_\alpha v) + k_\alpha B(v, \alpha^\vee)t_{s_\alpha}.$$

Subtract this identity from

$$t_{s_\alpha}vt_{s_\alpha} = s_\alpha v + k_\alpha B(v, \alpha^\vee)t_{s_\alpha}$$

to obtain $t_{s_\alpha}\tilde{v}t_{s_\alpha} = \widetilde{s_\alpha v}$. The assertion follows because the simple reflections generate W . $\square$

Proposition 4.2 (Hecke algebra anticommutator). *Let U be an $\mathbb{H}_J$ -module and put $E = \mathbb{H} \otimes_{\mathbb{H}_J} U$. For $H \in \mathfrak{a}_J$, define*

$$T_H(h \otimes u \otimes s) = h \otimes Hu \otimes s \quad (h \in \mathbb{H}, u \in U, s \in S_{\mathbb{H}}).$$

This is a well-defined endomorphism of $E \otimes S_{\mathbb{H}}$ commuting with $D_{\mathbb{H}}$. In the vector-space realization

$$E \simeq \mathbb{C}[W] \otimes_{\mathbb{C}[W_J]} U,$$

the formula

$$(4.4) \quad C_H(t_w \otimes u \otimes s) = t_w \otimes u \otimes c(wH)s$$

defines an endomorphism of $E \otimes S_{\mathbb{H}}$ satisfying

$$(4.5) \quad D_{\mathbb{H}}C_H + C_H D_{\mathbb{H}} = -2T_H.$$

If H acts invertibly on U , then $H_D(E) = 0$. In particular, if $H|_U = a \text{id}_U$ for $a \neq 0$, then $D_{\mathbb{H}}C_H + C_H D_{\mathbb{H}} = -2a \text{id}$ and $H_D(E) = 0$.

Proof. Since H is central in $\mathbb{H}_J$, the endomorphism $u \mapsto Hu$ is $\mathbb{H}_J$ -linear. It therefore induces an $\mathbb{H}$ -linear endomorphism of E . Acting trivially on the spin factor, this endomorphism is T_H , which consequently commutes with $D_{\mathbb{H}}$. The stated realization follows from the Poincaré–Birkhoff–Witt decomposition. If $j \in W_J$, then

$$t_{wj} \otimes u = t_w \otimes t_j u, \quad wjH = wH.$$

Consequently (4.4) respects the relations in the balanced tensor product and is well defined.

Lemma 4.1 gives, for every $w \in W$,

$$(4.6) \quad D_{\mathbb{H}}(t_w \otimes u \otimes s) = \sum_i t_w \otimes e_i u \otimes c(we_i)s - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha t_{ws_\alpha} \otimes u \otimes c(w\alpha^\vee)s.$$

Indeed, use the orthonormal basis (e_i) in (4.3), move t_w past each modified generator, and expand (4.2).

Apply C_H on the left and on the right of $D_{\mathbb{H}}$ in (4.6). Then the first summand of (4.6) becomes

$$\begin{aligned} \sum_i t_w \otimes e_i u \otimes (c(we_i)c(wH) + c(wH)c(we_i))s &= -2 \sum_i B(e_i, H)t_w \otimes e_i u \otimes s \\ &= -2t_w \otimes Hu \otimes s = -2T_H(t_w \otimes u \otimes s). \end{aligned}$$

Now study the second summand of (4.6):

$$-\frac{1}{2} \sum_{\alpha \in R^+} k_\alpha t_{ws_\alpha} \otimes u \otimes (c(w\alpha^\vee)c(wH) + c(ws_\alpha H)c(w\alpha^\vee))s.$$

We claim that

$$(4.7) \quad c(w\alpha^\vee)c(wH) + c(ws_\alpha H)c(w\alpha^\vee) = 0.$$

To verify the assertion, first observe that

$$s_\alpha H = H - B(H, \alpha^\vee)\alpha, \quad c(\alpha)c(\alpha^\vee) = -2\text{id}.$$

It follows that

$$c(\alpha^\vee)c(H) + c(s_\alpha H)c(\alpha^\vee) = -2B(\alpha^\vee, H)\text{id} - B(H, \alpha^\vee)c(\alpha)c(\alpha^\vee) = 0.$$

Since w preserves B , the same equality after applying w proves the vanishing of (4.7). Thus every reflection term cancels separately, proving (4.5). If $H|_U$ is invertible, its inverse is $\mathbb{H}_J$ -linear. In the stated vector-space realization,

$$T_H^{-1} = \text{id}_{\mathbb{C}[W]} \otimes (H|_U)^{-1} \otimes \text{id}_{S_{\mathbb{H}}}.$$

Lemma 2.1 applies. Explicitly,

$$x = D_{\mathbb{H}} \left(-\frac{1}{2} C_H T_H^{-1} x \right) \quad (x \in \ker D_{\mathbb{H}}).$$

The scalar case follows by taking $T_H = a\text{id}$. $\square$

4.3. Vanishing for a nonzero inducing character. For $J \subseteq \Pi$, put $V_J = \text{span}_{\mathbb{C}} J$. Then $V = V_J \oplus \mathfrak{a}_J$ orthogonally. Let $\mathbb{H}_J^{\text{ss}}$ be generated by $S(V_J)$ and $\mathbb{C}[W_J]$. The cross relations and the Poincaré–Birkhoff–Witt decomposition give

$$\mathbb{H}_J \simeq \mathbb{H}_J^{\text{ss}} \otimes S(\mathfrak{a}_J).$$

For $\nu \in \mathfrak{a}_J^*$, let $\mathbb{C}_\nu$ be the one-dimensional $S(\mathfrak{a}_J)$ -module on which H acts by $\nu(H)$. For an $\mathbb{H}_J^{\text{ss}}$ -module σ , put

$$E(J, \sigma, \nu) = \mathbb{H} \otimes_{\mathbb{H}_J} (\sigma \otimes \mathbb{C}_\nu).$$

Corollary 4.3 (Nonzero inducing character). *For every $J \subseteq \Pi$, every $\mathbb{H}_J^{\text{ss}}$ -module σ , and every $\nu \in \mathfrak{a}_J^* \setminus \{0\}$,*

$$H_D(E(J, \sigma, \nu)) = 0.$$

The parameter function k may be complex-valued.

Proof. Choose $H \in \mathfrak{a}_J$ with $\nu(H) \neq 0$. On $\sigma \otimes \mathbb{C}_\nu$, the element H acts by the scalar $\nu(H)$. Proposition 4.2 gives

$$D_{\mathbb{H}} C_H + C_H D_{\mathbb{H}} = -2\nu(H)\text{id}.$$

Lemma 2.1 proves the assertion. $\square$

Definition 4.4. Assume that the parameter function $k : R \rightarrow \mathbb{R}$ is real-valued. For $J \subseteq \Pi$, let $\omega_\alpha^J \in V_J$ be characterized by $B(\omega_\alpha^J, \beta^\vee) = \delta_{\alpha\beta}$ for $\alpha, \beta \in J$. A finite-dimensional $\mathbb{H}_J^{\text{ss}}$ -module σ is *tempered* if every generalized $S(V_J)$ -weight $\eta \in V_J^*$ of σ satisfies

$$\text{Re } \eta(\omega_\alpha^J) \leq 0 \quad (\alpha \in J).$$

A *Langlands standard module* is a module $E(J, \sigma, \nu)$, where σ is irreducible tempered and $\nu \in \mathfrak{a}_J^*$, extended by zero on V_J , satisfies

$$\text{Re } \nu(\alpha) > 0 \quad (\alpha \in \Pi \setminus J).$$

This is the convention of [11, §3 and Theorem 3.3].

Corollary 4.5. *Assume that k is real-valued, and let $E = E(J, \sigma, \nu)$ be a Langlands standard module as in Definition 4.4. If $J \neq \Pi$, then $H_D(E) = 0$. Consequently every non-tempered Langlands standard module has zero Dirac cohomology.*

Proof. If $J \neq \Pi$, the strict chamber condition implies $\nu \neq 0$, so Corollary 4.3 applies. If $J = \Pi$, then $\mathfrak{a}_J = V^W = 0$, and $E = \sigma$ is tempered by definition. $\square$

Remark 4.6. For crystallographic R , the ordinary Dirac index defined in [5, §2.5] vanishes for every finite-dimensional module induced from a proper parabolic subalgebra; see [5, §2.10 and Remark 3.1]. This is an equality of virtual representations, and does not by itself imply vanishing of Dirac cohomology. Corollary 4.3 proves the latter vanishing when $\nu \neq 0$.

4.4. Geometric realizations and parabolic induction. We apply Proposition 4.2 to the geometric standard modules of [4, Definition 4.6]. We retain the hypothesis that the roots span V . Note that our parametrization differs from that of [4, §4.1] by a scalar, which does not affect the vanishing assertion.

Let $\mathcal{G}$ denote the connected complex reductive group defining the geometric parameters, and let $\mathfrak{h}_{\text{geom}}$ be the Lie algebra of the connected centre of the cuspidal Levi subgroup. Thus $V = \mathfrak{h}_{\text{geom}}^*$. Here e is nilpotent and s is semisimple in the Lie algebra of $\mathcal{G}$, $[s, e] = 2re$, and ζ is an irreducible representation of the component group specified there.

Explicitly, put

$$\begin{aligned}\mathcal{Z}_{\mathcal{G}}(e, s) &= \{(g, a) \in \mathcal{G} \times \mathbb{C}^\times : \text{Ad}(g)e = a^2e, \text{Ad}(g)s = s\}, \\ A_{\mathcal{G}}(e, s) &= \mathcal{Z}_{\mathcal{G}}(e, s) / \mathcal{Z}_{\mathcal{G}}(e, s)^\circ.\end{aligned}$$

Here the superscript $\circ$ denotes the identity component, so $A_{\mathcal{G}}(e, s)$ is a finite group. Its action on the geometric module $E_{s,e,r}$ commutes with the $\mathbb{H}$ -action, and ζ is an irreducible complex representation of this group. The module with parameter ζ is the multiplicity space

$$E_{s,e,r,\zeta} = \text{Hom}_{A_{\mathcal{G}}(e,s)}(\zeta, E_{s,e,r}),$$

as in [4, Definition 4.6]. We write $E_{s,e,r,\zeta}^{\mathcal{G}}$ when it is necessary to specify the ambient group.

Throughout this subsection, assume $r \in \mathbb{R} \setminus \{0\}$. Let $\omega_\alpha \in V$ be the fundamental weights, characterized by $B(\omega_\alpha, \beta^\vee) = \delta_{\alpha\beta}$. Following [4, Definition 4.5], $E_{s,e,r,\zeta}$ is *tempered* if every generalized $S(V)$ -weight $\lambda \in V^*$ of $E_{s,e,r,\zeta}$ satisfies

$$\frac{\text{Re } \lambda(\omega_\alpha)}{r} \geq 0 \quad (\alpha \in \Pi).$$

Proposition 4.7. *Let $E = E_{s,e,r,\zeta}^{\mathcal{G}}$ be a nonzero geometric standard module which is not tempered. Choose an $\mathfrak{sl}_2$ -triple (e, h_e, f_e) satisfying $[s, h_e] = 0$ and $[s, f_e] = -2rf_e$, and put $z = s - rh_e$. For $c \in \mathbb{C}$, set*

$$\mathfrak{g}_{\text{geom}}(c) = \{x \in \text{Lie}(\mathcal{G}) : [z, x] = cx\}.$$

Let Q be the parabolic subgroup with Levi subgroup $\mathcal{M}$ defined by

$$(4.8) \quad \text{Lie}(Q) = \bigoplus_{\text{Re}(c)/r \leq 0} \mathfrak{g}_{\text{geom}}(c), \quad \text{Lie}(\mathcal{M}) = \bigoplus_{\text{Re}(c)/r = 0} \mathfrak{g}_{\text{geom}}(c).$$

After choosing a compatible positive system, the inherited cuspidal datum identifies the Hecke algebra of $\mathcal{M}$ with $\mathbb{H}_J$ for some $J \subsetneq \Pi$. Let $\zeta_{\mathcal{M}}$ correspond to ζ under the natural isomorphism of the component groups for (e, s) in $\mathcal{M}$ and $\mathcal{G}$. Then the geometric standard module for $\mathcal{M}$ with the same s, e, r satisfies

$$U := E_{s,e,r,\zeta_{\mathcal{M}}}^{\mathcal{M}}, \quad E_{s,e,r,\zeta}^{\mathcal{G}} \simeq \mathbb{H} \otimes_{\mathbb{H}_J} U.$$

Moreover, U is irreducible and has the form

$$U \simeq \sigma \otimes \mathbb{C}_\nu, \quad \nu \in \mathfrak{a}_J^* \setminus \{0\},$$

where σ is an irreducible $\mathbb{H}_J^{\text{ss}}$ -module. In particular, some $H_0 \in \mathfrak{a}_J$ acts invertibly on U .

Proof. The compatible triple and the parabolic subgroup in (4.8) are constructed in [10, §§3.11–3.12]. For this Levi subgroup, Lusztig's induction theorem gives induction from the nonzero irreducible module U , as recalled in [4, §5.1]. The identification of the component groups and the formula $U = E_{s,e,r,\zeta_{\mathcal{M}}}^{\mathcal{M}}$ are supplied by [10, §3.38(c),(e) and §3.39]; the inherited algebra is the full parabolic algebra $\mathbb{H}_J$ by [10, §1.16]. Since $S(\mathfrak{a}_J)$ is central in $\mathbb{H}_J$, Schur's lemma gives $U \simeq \sigma \otimes \mathbb{C}_\nu$ with σ irreducible. It remains to prove that $J \neq \Pi$ and $\nu \neq 0$.

Choose a parabolic subgroup P_0 in the fixed cuspidal conjugacy class such that $P_0 \subseteq Q$ and $z \in \mathfrak{p}_0 = \text{Lie}(P_0)$, and put

$$t = z \bmod [\mathfrak{p}_0, \mathfrak{p}_0] \in \mathfrak{p}_0 / [\mathfrak{p}_0, \mathfrak{p}_0] \simeq \mathfrak{h}_{\text{geom}}.$$

Such P_0 exists, and t is independent of its choice, by [10, Lemma 3.25]. Lemma 3.26(a),(b) gives

$$(4.9) \quad \frac{\operatorname{Re} \alpha(t)}{r} \begin{cases} = 0, & \alpha \in J, \\ < 0, & \alpha \in \Pi \setminus J. \end{cases}$$

Moreover, Lemma 3.26(c),(d), together with the weight identifications in [10, §3.36(c),(d) and §3.38(d)], gives, for every generalized $S(V)$ -weight η of U ,

$$(4.10) \quad \frac{\operatorname{Re} \eta(\omega_\alpha)}{r} \geq \frac{\operatorname{Re} \omega_\alpha(t)}{r} \quad (\alpha \in \Pi),$$

with equality for $\alpha \notin J$. These statements use our fundamental weights, which are positive multiples of Lusztig's dominant generators [10, §§3.4–3.6].

If $J = \Pi$, then $Q = \mathcal{G}$ and $U = E$. Since the roots span V , (4.9) implies $\operatorname{Re} \omega_\alpha(t)/r = 0$ for every α . Equation (4.10) would then make E tempered. Thus $J \neq \Pi$.

Choose $\beta \in \Pi \setminus J$. Then $H_0 = \omega_\beta \in \mathfrak{a}_J$, and every weight of U restricts to ν on $\mathfrak{a}_J$. Write

$$\omega_\beta = \sum_{\alpha \in \Pi} b_\alpha \alpha, \quad b_\alpha \geq 0, \quad b_\beta > 0,$$

using positivity of the inverse Cartan matrix on each irreducible component. The equality case of (4.10) and (4.9) yield

$$\frac{\operatorname{Re} \nu(H_0)}{r} = \frac{\operatorname{Re} \omega_\beta(t)}{r} = \sum_{\alpha \in \Pi} b_\alpha \frac{\operatorname{Re} \alpha(t)}{r} < 0.$$

The β -summand is strictly negative and all other summands are nonpositive. Hence $\nu \neq 0$ and $H_0|_U = \nu(H_0) \operatorname{id}_U$ is invertible. $\square$

Corollary 4.8 (Non-tempered geometric standard modules). *For every non-tempered geometric standard module E ,*

$$H_D(E) = 0.$$

Proof. Apply Proposition 4.2 to the induction description in Proposition 4.7. $\square$

Remark 4.9. Corollary 4.8 gives an anticommutator proof of the non-tempered vanishing in [4, Theorem 6.4]. The tempered results and the twisted-index criterion are [4, Theorems 6.2–6.3 and Corollary 6.6].

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